Electromagnetism and special relativity

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Fundamental basis of special relativity

During the last part of the 19th century, after Maxwell's equations were formulated (1862), many of their consequences were experimentally proved. Chief among these consequences is the propagation of electromagnetic waves, and thus of light, at a constant velocity c

$$c = 1/\sqrt{\varepsilon_0 \mu_0} = 2.99729\ 10^8\ m/s$$

in free space.

The existence of a specific propagation velocity of light in free space posed theoretical problems for contemporary physicists. This was because, according to Galileo's law of composition of velocities, the value of c must be valid only for a specific or preferential inertial system,[1] whereas in any other frame of reference the velocity of light must be different from c and must depend on the velocity of the observer. Furthermore, for those physicists a wave of any kind could exist only if there were a medium to support it. For this reason a hypothetical medium, termed ether, was assumed to fill all space and to be the preferential reference system in which the velocity of electromagnetic waves is c. This ether was assumed to have properties such that its density and interaction with matter were negligible while at the same time possessing a degree of rigidity great enough to enable a propagation velocity as high as that of electromagnetic waves. Furthermore, because ether could act as a preferential system to measure the absolute velocity of any other referential system, electromagnetism was located on a different plane from all other physical phenomena then known. This was because, since Newton's laws of mechanics were invariant against the universally accepted Galileo transformations between inertial systems, there could be no preferential inertial system to determine the absolute speed of another reference.intense experimental activity took place to determine

[1] Se denomina sistema inercial aquel en el que un cuerpo sobre el que no actúa fuerza alguna, se encuentra en movimiento rectilíneo uniforme o bien en reposo. There is an infinite number of inertial frames es infinito since cualquier sistema que se mueva con velocidad rectilínea uniforme respecto a un sistema inercial dado, es también inercial. En lo que sigue los sistemas de coordenadas serán siempre inerciales.

the existence of such an ether, to measure the velocity of the earth with respect to it, to test whether Maxwell's equations were correctly formulated and to determine whether they were invariant or not under a Galilean transformation.

The most famous attempt was the Michelson-Morley experiment to observe certain properties of the ether and the motion of the earth through it. Bassically the goal of the experiment was to determine the difference between light travelling parallel to the earth's trajectory and light passing in a perpendicular direction. The experiment was simple and precise though presenting a basic contradiction to Galilean-Newtonian kinematiks.

From all this activity, the following conclusions were drawn:

- Maxwell's equations, which predict that electromagnetic waves move at speed c regardless of the frame of reference, were found to be well formulated and consequently the Galileo transformations proved incompatible with Maxwell's equations

- The Galileo transformations between inertial systems are a good approximation of the correct ones, termed the Lorentz transforms, which simplify to the former only when two inertial systems move with respect to each other at velocity $v << c$.

- Electromagnetic waves are self-sustaining and need no support of a physical medium to propagate; thus they can propagate in a vacuum. Therefore, the supposed ether does not exist. The propagation velocity of electromagnetic waves is $c = 1/\sqrt{\varepsilon_0 \mu_0}$ and is identical for any inertial system, irrespective of the relative velocities of the source or of the observer [2].

These conclusions were presented in 1905 by Einstein, who formulated the fundamental principle of the special theory of relativity, which states:

All physical laws and the results of any experiment are the same in all inertial systems. In other words, it is impossible to perform a physical experiment which differentiates in any fundamental sense between different inertial frames.

It is worth noting that from this principle the constancy of c can be deduced directly because ε_0 and μ_0 are physical constants that can be easily evaluated by means of two simple experiments: ε_0 can be measured from the interaction force between two fixed charges, and μ_0 from the interaction between two parallel conducting wires, each bearing a current intensity. From the fundamental principle of relativity, ε_0 and μ_0, and therefore c, must have the same value in all inertial systems.

0.1 Lorentz transformations

To deduce the expressions that transform the space and time coordinates of one inertial system into another, we start from the postulates stated in the previous section, assuming the homogeneity and isotropy of the space. Let us assume

[2] Indeed, any wave-like disturbance which is self-sustaining and does not require a medium through which to propagate such as a gravity wave, should travel at the same velocity in all inertial frames. Otherwise we could differenciate inertial frames using the apparent propagation speed of the disturbance, which would violate the relativity principle.

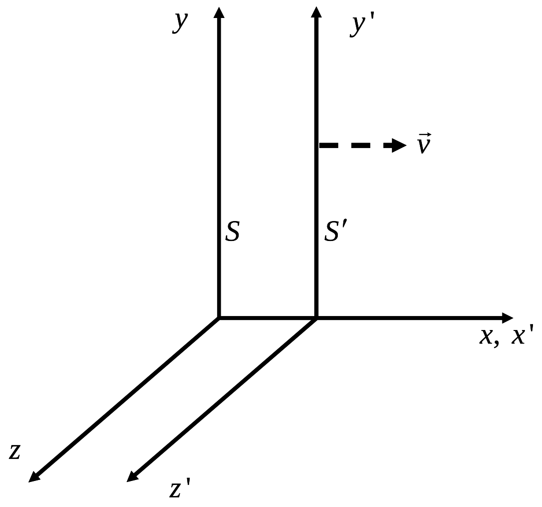

Figure 1:

two inertial systems in the standard configuration, S and S' with a common axis (x , x') and that S' is moving in the x-direction at a uniform velocity $\vec{v}$ while maintaining the axes y , y' and z , z' parallel, as shown in Figure 1.

Let us consider an event in the inertial reference S and determine its position and time by specifying its coordinates x, y, z, t. In a second inertial frame S' the same event is recorded with the space and time coordinates x', y', z', t'. Now we seek to deduce the expressions

$$x' = x'(x, y, z, t) \tag{1a}$$
$$y' = y'(x, y, z, t) \tag{1b}$$
$$z' = z'(x, y, z, t) \tag{1c}$$
$$t' = t'(x, y, z, t) \tag{1d}$$

which enable us to relate the space and time coordinates measured by an observer of the event at S' to those measured by another observer of the same event at S.

The functional relations (1) can be found from the fundamental principle of the special theory of relativity, assuming that space and time are homogeneous. This homogeneity means that the transformation relations are linear, and consequently the measurement of a length or a time interval between two events in a given reference system does not depend on when or where the measurement is made. Therefore, the most general form that the equations (1) may take is

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \tag{2a}$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t \tag{2b}$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t \tag{2c}$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \tag{2d}$$

To obtain the values of the 16 coefficients $a_{\alpha\beta}$ ($\alpha,\ \beta = 1,2,3,4$) in (2), let us assume that they may depend on the relative velocity v between S and S' and that, at the instant at which the origins O of S and O' of S' coincide, we set the clocks to show $t = 0$ and $t' = 0$, respectively. In other words, we assume that the events $(0,0,0,0)$ coincide in both frames.

As there is no relative movement between the systems in the y and z directions, we assume that $y' = y$, $z' = z$, such that

$$a_{22} = a_{33} = 1 \tag{3}$$
$$a_{21} = a_{23} = a_{24} = a_{31} = a_{32} = a_{34} = 0 \tag{4}$$

Let us now analyze *Eq.* (2d). Assuming space to be isotropic, we see that t' cannot depend on y or on z, since in such a case two clocks situated symmetrically in the plane $x = 0$ would disagree when observed from S'. Therefore

$$a_{42} = a_{43} = 0 \tag{5}$$

With respect to x', we know that a point where $x' = 0$ seems to move in the positive direction of the x axis at a velocity v, and so the proposition $x' = 0$ must be equivalent to $x = vt$. In other words, the relation must be of the type $x' = a_{11}(x - vt)$, and hence we get

$$a_{14} = -va_{11} \tag{6}$$

Thus (2) simplifies to

$$x' = a_{11}(x - vt) \tag{7a}$$
$$y' = y \tag{7b}$$
$$z' = z \tag{7c}$$
$$t' = a_{41}x + a_{44}t \tag{7d}$$

To determine the coefficients a_{11}, a_{41} and a_{44}, let us assume that in $t = t' = 0$, when the two coordinate origins O and O' coincide, an electromagnetic wave is created at the two origins. According to the theory of relativity this wave propagates in both inertial systems at a velocity c in all directions. The wave front of each system is given by

$$x^2 + y^2 + z^2 = c^2t^2 \tag{8}$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2 \tag{9}$$

By substituting (7) in (9) we get

$$a_{11}^2 (x - vt)^2 + y^2 + z^2 = c^2 (a_{41}x + a_{44}t)^2 \tag{10}$$

and thus

$$\left(a_{11}^2 - c^2 a_{41}^2\right) x^2 + y^2 + z^2 - 2\left(v a_{11}^2 + c^2 a_{41} a_{44}\right) xt = \left(c^2 a_{44}^2 - v^2 a_{11}^2\right) t^2 \tag{11}$$

For this expression to be consistent with (8) the coefficients must verify

$$\begin{aligned} a_{11}^2 - c^2 a_{41}^2 &= 1 \\ v a_{11}^2 + c^2 a_{41} a_{44} &= 0 \\ c^2 a_{44}^2 - v^2 a_{11}^2 &= c^2 \end{aligned} \tag{12}$$

which is a three-equation system with three unknowns, the solutions to which are

$$a_{44} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \; ; \; a_{11} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \; ; \; a_{41} = -\frac{v}{c^2\sqrt{1 - \frac{v^2}{c^2}}} \tag{13}$$

By substituting (13) in (7) we get the relativist transformation equations of the space and time coordinates

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{14a}$$

$$y' = y \tag{14b}$$

$$z' = z \tag{14c}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{14d}$$

which are known as the *Lorentz transformation.*

To simplify the notation, it is customary to define

$$\beta = \frac{v}{c} \tag{15}$$

$$\gamma = \left(1 - \beta^2\right)^{-1/2} \tag{16}$$

and so Eqs. (14) simplify to

$$x' = \gamma(x - vt) \tag{17a}$$

$$y' = y \tag{17b}$$

$$z' = z \tag{17c}$$

$$ct' = \gamma(ct - \beta x) \tag{17d}$$

The inverse Lorentz transformation, i.e., a Lorentz transformation in which the velocity of the moving frame is $-v$ along the x-axis instead of v, is performed by substituting v by $-v$ in (17). Note that when v/c tends to zero, the Lorentz transformation tends to the Galilean expressions

$$x' = x - vt; \quad y' = y; \quad z' = z; \quad t' = t \tag{18}$$

For $v > c$ the coordinates x', t' are imaginary. This signifies that a velocity greater than that of light is impossible. It is not even possible to use a reference system that moves at the velocity of light, since the denominators that appear in (17a) and (14d) would be null. It can be shown that the Lorentz transformation leaves the wave equation unchanged.

0.2 Consequences of the Lorentz transformations. Spatial and temporal intervals.

The Lorentz transforms lead to consequences that contradict standard notions regarding the properties of space and time, based on everyday experience. For example, from these transforms we deduce that concepts such as the size of a body or the time interval between two physical phenomena are not absolute in character and differ according to the reference system.

Let us consider, first, the concept of spatial extension, or length. Let us assume that in a system S' a body, which we term scale, is at rest. No force that might deform or modify its size is acting on this body. The length of the scale in the direction of movement (the x axis) is designated by l_0. This length, measured in system S', is called the proper length of the scale. Let x'_1 and x'_2 be the coordinates of the origin and the end of the scale in the S' reference system such that $x'_2 - x'_1 = l_0$. To measure the length of the scale in system S we must fix the coordinates of its origin, x_1, and of its end, x_2, at the same instant t. By means of (14a), we have

$$x'_2 = (x_2 - vt)\gamma \tag{19}$$

$$x'_1 = (x_1 - vt)\,\gamma \tag{20}$$

and therefore

$$l = \frac{l_0}{\gamma} \tag{21}$$

where $l = x_2 - x_1$.

We see, thus, that the length of a scale moving with velocity v with respect to system S is $1/\gamma$ times its proper length. This change in the dimensions of a body is usually called the Lorentz contraction. As the dimensions of the scale in the directions perpendicular to the velocity remain invariant, the volume of the scale is linked to its proper volume by the expression

$$V = \frac{V_0}{\gamma} \tag{22}$$

Therefore, the length and the volume of a scale that is not subject to the action of external forces have a relative value, and the distance between two points depends on the movement of the reference system. Of course if the scale is at rest in S, its length in S' is less than in S in the same proportion, as there is complete reciprocity between the two reference systems.

Relativity theory implies an equally fundamental change in the notion of time. Assume that at a point x' in the system S' a given physical process occurs during the time interval $\triangle t_0 = t'_2 - t'_1$, where t'_1 and t'_2 are the initial and final instants of the process. In system S for the corresponding instants t_1 and t_2 we can write

$$t_1 = \left(t'_1 + \frac{vx'}{c^2}\right)\gamma$$

$$t_2 = \left(t'_2 + \frac{vx'}{c^2}\right)\gamma$$

By subtraction, we find the time interval $\triangle t$ that has elapsed from the beginning to the end of the process in system S

$$\triangle t = t_2 - t_1 = \triangle t_0 \gamma \tag{23}$$

The time $\triangle t_0$ measured in the reference system that is moving in conjunction with the body in which the process occurs is called proper time.

The expression (23) shows that the proper time between two physical events is $1/\gamma$ times that which elapses between them *in* system S. Therefore, unlike what happens in Newtonian physics, the time interval depends on the state of movement. There is no universal time, and the concept of an interval of time between two successive physical events is a relative one. Here, too, there is complete reciprocity between systems S and S'[3].

0.3 Einstein's law of composition of velocities

To find the relativistic law of composition of velocities, we can write the Lorentz transformation formulas in differential form

$$dx = (dx' + vdt')\,\gamma \tag{24a}$$

$$dy = dy' \tag{24b}$$

$$dz = dz' \tag{24c}$$

$$dt = \left(dt' + \frac{vdx'}{c^2}\right)\gamma \tag{24d}$$

Let x', y', z' be the coordinates of a material point that moves within the reference system S'. The components of the velocity at this point in system S' will be

$$u'_x = \frac{dx'}{dt'}\ ,\ u'_y = \frac{dy'}{dt'},\ u'_z = \frac{dz'}{dt'} \tag{25}$$

[3] Aunque la teoría de la relatividad revolucionó los conceptos de la Física newtoniana acerca del carácter absoluto de espacio y tiempo, debemos tener en cuenta que el objetivo de la teoría de la relatividad es hallar las leyes absolutas de la naturaleza, independientes del sistema inercial de referencia lo que está ligado con encontrar magnitudes invariantes como por ejemplo la velocidad de la luz c .

and in system S

$$u_x = \frac{dx}{dt},\ u_y = \frac{dy}{dt},\ u_z = \frac{dz}{dt} \tag{26}$$

Hence, from (24) we have

$$u_x = \frac{dx}{dt} = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}},\ u_y = \frac{dy}{dt} = \frac{u'_y}{(1 + \frac{u'_x v}{c^2})\gamma},\ u_z = \frac{dz}{dt} = \frac{u'_z}{(1 + \frac{u'_x v}{c^2})\gamma} \tag{27}$$

The above expressions constitute what is known as Einstein's law of composition of velocities and replace the formulas given by classical mechanical theory Galileo's velocity composition law,

$$u_x = u'_x + v,\ u_y = u'_y,\ u_z = u'_z \tag{28}$$

which are an approximation of Einstein's law for $v << c$.

From expressions (27) we see again, as expected, that the velocity of light is a limit velocity. If, for example, a particle moves in system S' along axis x at a velocity $u' = u'_x = c$, in the resting system S its velocity will be

$$u = \frac{c + v}{1 + \frac{cv}{c^2}} = c \tag{29}$$

If the velocity of the particle in S' is less than that of light, for example

$$u' = u'_x = c - a \quad (a > 0) \tag{30}$$

and system S' is moving with respect to S at a velocity $v = c - b$; $b > 0$, then the velocity of the particle with respect to system S is equal to

$$u = \frac{(2c - a - b)\,c}{2c - a - b + \frac{ab}{c}} < c \tag{31}$$

Thus, the sum of two velocities, each of which is less than that of light, is always less than the velocity of light. The sum of two velocities, one of which is equal to the velocity of light while the other is less, is equal to the velocity of light.

From the composition of velocities, it follows that an angle has a relative value i.e., it depends on the inertial reference system. Given that

$$\tan\theta = \frac{u_y}{u_x}$$

where θ is the angle formed by the velocity vector of the particle and by the x axis. From (27) it follows that

$$\tan\theta = \frac{u'\sin\theta'}{\left(v + u'\cos\theta'\right)\gamma} \tag{32}$$

where $u'_x = u'\cos\theta'$, $u'_y = u'\sin\theta'$. This formula expresses the law of angle transformation as part of the theory of relativity. This law links the values of the angles θ and θ' formed by the velocity vector with the x' and x axes, respectively.

Finally, it should be noted that when we use the expression 'velocity of a body', we refer to the velocity with which a real body moves or with which a real process of interaction (i.e. a signal) is propagated. Without contradicting the theory of relativity, we can imagine processes in which the velocity is greater than the velocity of light, but these are spatial in nature and are incapable of transporting a body or of resolving an interaction.

0.4 Simultaneity

Let us assume that in an inertial reference system S' two physical events occur simultaneously at points x'_1 and x'_2 at a given instant t'. In the inertial system S the first event occurs at instant

$$t_1 = \left(t' + \frac{x'_1 v}{c^2}\right)\gamma$$

and the second, at

$$t_2 = \left(t' + \frac{x'_2 v}{c^2}\right)\gamma$$

Therefore, in S these two events do not occur at the same instant, but after a given time $\triangle t$ has elapsed

$$\triangle t = t_2 - t_1 = \frac{v}{c^2}\left(x'_2 - x'_1\right)\gamma \tag{33}$$

Moreover, depending on the sign of $(x'_2 - x'_1)$, the time interval $\triangle t$ may be positive or negative, i.e. in S the "first" event occurs before or after the "second" one. Thus, the concept of simultaneity is relative. The only exception to this, although a very important one, arises when the two events occur simultaneously at the same point. In this case, from (33), in every reference system (and at any velocity v) it is verified that $\triangle t = 0$ i.e., simultaneity is absolute.

0.5 Geometric representation of the Lorentz transformations

To provide a formal mathematical structure to the relativistic expressions, it is very helpful to use a 4-dimensional space in which the axes are the three spatial coordinates and where, as suggested by Minkowski, ct, instead of t alone, is the fourth coordinate. Thus, a point (called world point) is defined by the coordinates (ct, x, y, z), which at the same time determine the radius vector from the origin to the point. The Lorentz transformation and its consequences can be interpreted geometrically using the Minkowski diagram, which provides a geometric representation in the 4-dimensional space of the evolution of an

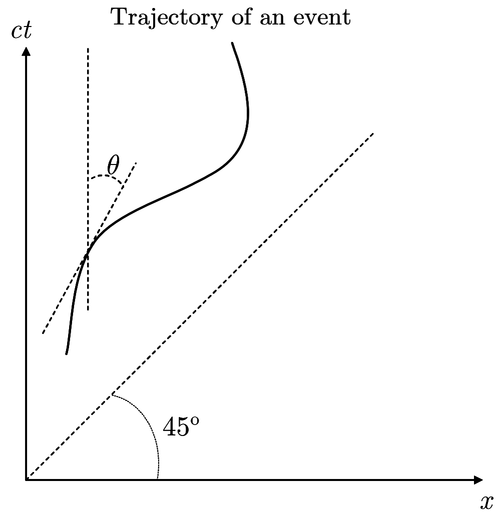

Figure 2: aqui la curva sobrepasa los 45^o hacer la curva menos abierta

event. For example, in Figure 2 the trajectory of a particle appears as a line, called the world line. To represent the Minkowski diagram on a single plane this figure does not show the y and z coordinates since they are not affected by the Lorentz transformation. The tangent of the angle θ that the trajectory forms with the time axis is

$$\theta = \arctan\left(\frac{dx}{cdt}\right) = \arctan\left(\frac{u}{c}\right) \tag{34}$$

where u is the instantaneous velocity of the particle. Because it is not possible to travel faster than light, the angle of inclination of this curve with respect to the time axis is always less than 45°. The movement diagram for an electromagnetic wave or a photon travelling at the velocity of light is a 45° line. The lines $x = \pm ct$ divide the Minkowski diagram into regions as shown in Figure 3. At $t = 0$, the upper unshaded region, called the future, represents the region containing the points that the particle is able to reach at times $t > 0$. The lower unshaded region, called the past, represents the points that the particle could already have passed in its trajectory. In a three-dimensional plot, the lines $x = \pm ct$ become light cones whose surfaces are spheres defined by $x^2 + y^2 + z^2 = (ct)^2$.

F*igure 4 shows the axes of the system* S' plotted in the Minkowski diagram of S. Given that point $x' = 0$ is moving with respect to S at a velocity v its world line is a straight line that passes through the origin and, together with the time axis, forms an angle $\theta = \arctan\beta$. This line verifies that $x' = 0$ and so coincides with the time axis of system S'. By making $ct' = 0$ in (17d) we obtain the equation $ct = \beta x$ of the spatial axis x' that corresponds to a straight line forming an angle $\theta = \arctan\beta$ with the x axis. In general, in this figure

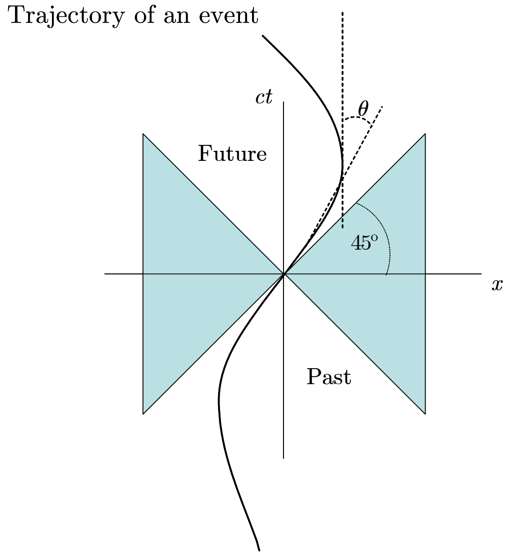

Figure 3:

the lines $t = cte$ are parallel to the Ox axis, while the lines $t' = cte$ are parallel to Ox'. Figure *4* shows the relative nature of the simultaneity; all the events occurring on axis x' are simultaneous in system S' but not in S. For example, for S the event given by point A', which in S' is simultaneous with event O, occurs a certain time later, $t = AA'/c$, than the event defined by point O.

0.5.1 The invariant interval

Let us assume that an event 1 occurs at (x_1, y_1, z_1, ct_1) and an event 2 at (x_2, y_2, z_2, ct_2) in the Minkowski diagram. The space-time interval s_{12} between the two events is defined as

$$s_{12}^2 = c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = c^2(t_2 - t_1)^2 - r_{12}^2 \quad (35)$$

where r_{12} is the spatial distance between the two events. For $x_2 = 0$, $y_2 = 0$, $z_2 = 0$, $t_2 = 0$, (35) simplifies to

$$s^2 = c^2t^2 - x^2 - y^2 - z^2 = c^2t^2 - r^2 \quad (36)$$

or in differential form

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt^2 - dr^2 \quad (37)$$

It is easy to see that the interval between two events is invariant under the Lorentz transformation, that is

$$\begin{aligned} s^2 &= c^2t^2 - x^2 - y^2 - z^2 = c^2t^2 - r^2 \\ &= s'^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t'^2 - r'^2 = I \end{aligned} \quad (38)$$

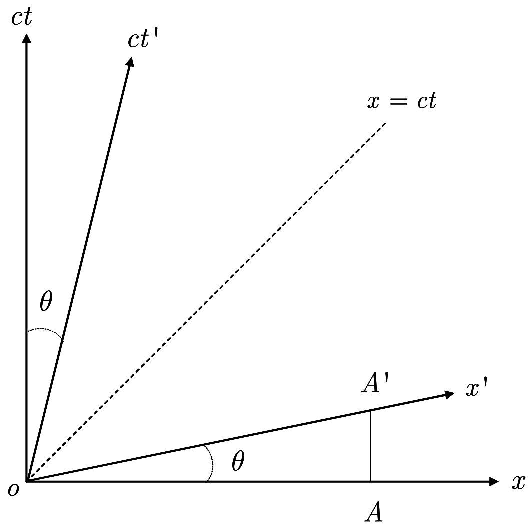

Figure 4: Ver pp 36 ruso new

We say that two events are **spacelike, lightlike** or **timelike** if the value of the interval between them is greater than, equal to or less than 0, respectively. If we connect the events by a straight line in the Minkowski diagram, these three cases correspond to a value of the slope of the line that is greater than, equal to or less than 1, respectively.

0.5.2 Proper time

Let us consider a particle moving along its world line with a velocity $u(t)$ and let S' be an instantaneous proper system in which the particle is at rest at the instant t. Let us assume that at the same point, x', y', z', in S' two events occur, separated by an infinitesimal time interval $d\tau$. This latter is called proper time because it is measured for a clock attached to the particle at rest in S'. Since there is no space interval between the two events, according to (37), the interval ds between them is

$$c^2 d\tau^2 = ds^2 \tag{39}$$

and

$$d\tau = \frac{ds}{c} \tag{40}$$

Hence, the proper time is an invariant because ds and c are invariant.

In an inertial coordinate system S, in which the particle is moving at a velocity $u(t)$, we have $d\vec{r} = \vec{u}dt$, such that[4]

$$ds^2 = c^2dt^2 - u^2dt^2 = c^2dt^2\left(1 - \frac{u^2}{c^2}\right) \tag{41}$$

and from (40)

$$d\tau = dt\sqrt{1 - \frac{u^2}{c^2}} \tag{42}$$

Thus the finite interval of the proper time τ is equal to

$$\tau = \int_0^{t_0} \left(1 - \frac{u^2}{c^2}\right)^{1/2} dt \tag{43}$$

It should be noted that this formula has been deduced for the case of the movement of clocks in conjunction with an inertial reference system, i.e., one in which the movement presents a constant velocity. Although it is frequently applied to an accelerated movement, in which v is considered to be a function of time, we should remember that the special theory of relativity does not allow an accelerated movement of the reference system. Therefore, in the case of an accelerated movement, proper time does not make sense. Nevertheless, it is a useful magnitude, one that is invariant with respect to the Lorentz transformations

For a complete representation of the Lorentz transformation, it is still necessary to specify the units on the axes. For this reason, Figure 23.3 shows the following two hyperbolas:

$$(x)^2 - (ct)^2 = \pm 1 \tag{44}$$

These cut the axes x^0 and x^1 of system S at points $(ct = 1, x = 0)$ and $(ct = 0, x = 1)$. The intersection points with the ct' and x'axes of system S' are $(ct' = 1, x' = 0)$ and $(ct' = 0, x' = 1)$, respectively. From Figure 23.5 (libro mio castellano) the phenomenon of the reciprocal intersection of scales can be understood by considering a unit scale OA that is at rest with respect to S. The lines of the universe at its end points are ODC and AA'.

For an observer at rest at point S', the location of the scale gives the world points O and A', respectively, for its origin and end points. Thus, for this observer the scale is shorter than the unit of length OB'. Reciprocally, the end and the origin of a unit of length OB' at rest in system S' with world lines $OC'D'$ and BB' appears for system S at $ct = 0$, which corresponds to $t = 0$, at the world points O and B. Once again, the scale is shorter than its unit of length.

Clocks may be compared in the same way. A clock at rest at point S' moves along the world line $OC'D'$. At the world point $(ct = 1)$ a unit of time is marked, but beforehand (at the world point C') the clock in system S that spatially coincides with the former has made one revolution $(ct = 1)$. Therefore, the clock that is moving marks time more slowly than the one that is at rest.

[4]Note that we use the simbol $\vec{u}$ for the velocity of a particular object and reserve $\vec{v}$ for the relative velocity of a particular coordinate system

0.6 4-dimensional vectors. Covariant and contravariant components

The coordinates of a world point can be written in a concise way, called contravariant form, as x^μ, where $\mu = 0, 1, 2, 3$, such that

$$x^\mu = (x^0, x^1, x^2, x^3) \tag{45}$$

and

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z \tag{46}$$

which transform from one coordinate system to another according to (17), that is

$$\begin{aligned} x'^0 &= \left(x^0 - \beta x^1\right)\gamma \\ x'^1 &= \left(x^1 - \beta x^0\right)\gamma \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned} \tag{47}$$

with

$$x'^\mu = (x'^0, x'^1, x'^2, x'^3) \tag{48}$$

By analogy, we define a contravariant 4-vector A^μ in the 4-dimensional space as any set of four quantities

$$A^\mu = (A^0, A^1, A^2, A^3) \tag{49}$$

which transform from one coordinate system to another as the coordinates of a world point, that is

$$\begin{aligned} A'^0 &= \left(A^0 - \beta A^1\right)\gamma \\ A'^1 &= \left(A^1 - \beta A^0\right)\gamma \\ A'^2 &= A^2 \\ A'^3 &= A^3 \end{aligned} \tag{50}$$

The first component, A^0, of a 4-vector is usually referred to as the time component of the 4-vector while the three following ones, A^1, A^2, A^3, are called the space components. When the space components of a 4-vector A^μ correspond to those of a given 3-vector $\vec{A}$, we can use the notation

$$A^\mu = (A^0, \vec{A}) \tag{51}$$

For example, the three last components of a world point, (45), are the components of the spatial position 3-vector $\vec{r}$ and we can write

$$x^\mu = (x^0, \vec{r}) \tag{52}$$

The covariant form, A_μ, of the 4-vector A^μ is defined as

$$A_\mu = (A_0, A_1, A_2, A_3) \tag{53}$$

where

$$A_0 = A^0,\ A_1 = -A^1,\ A_2 = -A^2,\ A_3 = -A^3 \tag{54}$$

Note that the covariant form A_μ of a 4-vector is obtained from the contravariant one A^μ (or vice versa) by simply changing the sign of the space components, and that we use upper and lower indexes to distinguish between the contravariant and the covariant forms of a 4-vector, respectively.

The scalar or inner product of two 4-vectors is defined as

$$A^\mu B_\mu \tag{55}$$

where we have made use of Einstein's summation convention that the repetition of an index automatically implies a summation over it,

$$A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \tag{56}$$

Vectors A^μ and B^μ are said to be orthogonal if $A^\mu B_\mu = 0$. The square of a 4-vector A^μ is its scalar product with itself, that is

$$A^\mu A_\mu \tag{57}$$

It is easy to show that (55), and consequently (57), are invariant under the Lorentz transformation. The *space-time* interval s^2 (36) between two events

$$s^2 = x^\mu x_\mu \tag{58}$$

is an example of an invariant corresponding to the square of a radius vector.

Using the summation convention, we can write expressions (50) in matrix form as

$$A'^\mu = \Lambda^\mu_\nu A^\nu \tag{59}$$

and

$$A'_\mu = \left(\Lambda^\mu_\nu\right)^{-1} A_\nu \tag{60}$$

respectively, where Λ^μ_ν represents the Lorentz transformation matrix defined by

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{61}$$

and $\left(\Lambda^\mu_\nu\right)^{-1}$ is given by the inverse matrix of Λ^μ_ν

$$\left(\Lambda^\mu_\nu\right)^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{62}$$

such that

$$\Lambda_{\alpha}^{\mu}\left(\Lambda_{\mu}^{\beta}\right)^{-1} = \delta_{\alpha}^{\beta} \tag{63}$$

with δ_{α}^{β} being the 4-dimensional version of the Kronecker delta or, in matrix form, the identity matrix defined by

$$\delta_{\alpha}^{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{64}$$

If we solve (59) for A^{ν} we have

$$A^{\nu} = \left(\Lambda_{\nu}^{\mu}\right)^{-1} A'^{\mu} \tag{65}$$

thus $\left(\Lambda_{\nu}^{\mu}\right)^{-1}$ is also the matrix of the transformation from S' to S.

0.7 4-Tensors

The principle of relativity requires that the mathematical equations expressing the laws of nature be covariant[5], i.e., invariant in form under the Lorentz transformation from one inertial system of coordinates to another. Tensor analysis is of primary importance in the theory of relativity, since tensor equations have the same form in all coordinate systems; for an equation to be covariant, it is necessary and sufficient that it can be expressed in tensor form.

In general, a tensor of rank r in an n-dimensional space has n^r components which are usually a function of position in that space. However, in special relativity we are concerned almost exclusively with the Lorentz transformation in the 4-dimensional space, with tensors no higher than second rank and with constant values for the coefficients of the Lorentz transformation matrix Λ_{β}^{α}. Thus, in the following, these simplifications will be generally assumed.

A 4-tensor of rank 2 is an array of 16 quantities which are transformed from the coordinate system S to S' as

$$T'^{\mu\nu} = \Lambda_{\alpha}^{\mu}\Lambda_{\beta}^{\nu}T^{\alpha\beta} \quad \text{(contravariant tensor)} \tag{66a}$$

$$T'_{\mu\nu} = \left(\Lambda_{\alpha}^{\mu}\right)^{-1}\left(\Lambda_{\beta}^{\nu}\right)^{-1}T_{\alpha\beta} \quad \text{(covariant tensor)} \tag{66b}$$

$$T'^{\mu}_{\nu} = \Lambda_{\alpha}^{\mu}\left(\Lambda_{\beta}^{\nu}\right)^{-1}T^{\alpha}_{\beta} \text{ or } T'^{\nu}_{\mu} = \left(\Lambda_{\alpha}^{\mu}\right)^{-1}\Lambda_{\beta}^{\nu}T^{\beta}_{\alpha} \quad \text{(mixed tensor)} \tag{66c}$$

where we use prime to distinguish between tensors in different coordinate systems, and represent a tensor by a typical component with indexes ranging from

[5] Notice that, as an universally accepted convention, the term covariant is used in two different ways. When we say that an expression is covariant we mean that its form does not change under a Lorentz transformation. When we say that a vector is covariant we mean that transforms according to (50) and (54).

0 to 3. Tensors can be contravariant, $T^{\mu\nu}$, covariant $T_{\mu\nu}$ or mixed T_{μ}^{ν}. The number of indexes determines the rank of the vector. For example, $T^{\mu\nu}$, $T_{\mu\nu}$ and T_{μ}^{ν} are tensors of rank 2, while a tensor such as $T^{\mu\nu\delta}$ would be a contravariant tensor of rank 3 with 64 components, which are transformed according to $T'^{\mu\nu\delta} = \Lambda_{\alpha}^{\mu}\Lambda_{\beta}^{\nu}\Lambda_{\sigma}^{\delta}T'^{\alpha\beta\sigma}$. In particular, a 4-vector is a tensor of rank 1 and a scalar is a tensor of rank zero. The latter is often termed an invariant which can be specified by a single number. An example of a scalar is the square of a vector (57).

The sum of two tensors $T_{\alpha\beta}$ and $P_{\alpha\beta}$ is defined as a tensor $Q_{\alpha\beta}$ of which the components are

$$Q_{\alpha\beta} = T_{\alpha\beta} + P_{\alpha\beta} \tag{67}$$

The difference between two tensors is defined in a similar way. The product of two tensors of rank n and r is a tensor of $(n+r)$ rank. Thus, a rank 2 tensor can be constructed from two 4-vectors; for example A^{μ} and B^{ν}, as $T^{\mu\nu} = A^{\mu}B^{\nu}$ such that $T'^{\mu\nu} = A'^{\mu}B'^{\nu} = \Lambda_{\alpha}^{\mu}A^{\alpha}\Lambda_{\beta}^{\nu}B^{\beta} = \Lambda_{\alpha}^{\mu}\Lambda_{\beta}^{\nu}A^{\alpha}B^{\beta} = \Lambda_{\alpha}^{\mu}\Lambda_{\beta}^{\nu}T^{\alpha\beta}$. Of course, we can repeat this for tensors of greater rank, for example $T_{\alpha}^{\nu}P_{\beta\gamma} = Q_{\alpha\beta\gamma}^{\nu}$. A symmetric tensor S_{ν}^{μ} is defined by the condition $S_{\nu}^{\mu} = S_{\mu}^{\nu}$ and an antisymmetric tensor A_{ν}^{μ} by the condition $A_{\nu}^{\mu} = -A_{\mu}^{\nu}$. An antisymmetric tensor has only six independent elements with its diagonal elements being zero while a symmetric tensor has ten independent elements. It is possible to express any tensor as the sum of a symmetric and an antisymmetric tensor, for example

$$T_{\nu}^{\mu} = \frac{1}{2}\left(T_{\nu}^{\mu} + T_{\mu}^{\nu}\right) + \frac{1}{2}\left(T_{\nu}^{\mu} - T_{\mu}^{\nu}\right) = S_{\nu}^{\mu} + A_{\nu}^{\mu} \tag{68}$$

where the first parenthesis contains a symmetric tensor and the second one contains an antisymmetric tensor.

The spacetime metric tensor $g_{\alpha\beta}$ is a very simple example of a tensor and is defined as [6]

$$g_{\alpha\beta} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{69}$$

It is easy to show the following relations between the contravariant, covariant and mixed forms of the metric tensor

[6] There are other possibilities of notation. For example, instead of our convention, called timelike, where the minus sign is associated with the spatial variables $(+,-,-,-)$, some authors use the spacelike convention where the minus sign is associated with the time variable $(-,+,+,+)$. In this case $g_{\alpha\beta}$ is defined as

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$$g_{\mu\nu} = g_{\nu\mu} \tag{70a}$$
$$g^{\mu\nu} = g_{\mu\nu} \tag{70b}$$
$$g_{\mu\delta}g^{\delta\nu} = g^{\nu}_{\mu} = \delta^{\nu}_{\mu} \tag{70c}$$

The metric tensor raises or lowers an index of a tensor, for example

$$A_{\mu} = g_{\mu\nu}A^{\nu} \quad ; \quad A^{\mu} = g^{\mu\nu}A_{\nu} \tag{71}$$

i.e. the covariant and contravariant form of a 4-vector can be interchanged using $g_{\mu\nu}$.

The scalar product (56) can be written as

$$A^{\nu}B_{\nu} = A_{\mu}B^{\mu} = g_{\mu\nu}A^{\nu}B^{\mu} \tag{72}$$

and the invariants (36) and (37) as

$$s^2 = g_{\mu\nu}x^{\mu}x^{\nu} \text{ or } ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{73}$$

respectively s^2 is the norm.

0.7.1 Differential operators

The 4-dimensional gradient or 4-gradient is defined as the operator

$$\frac{\partial}{\partial x^{\mu}} = (\frac{\partial}{\partial x^0}, \nabla) = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla) \tag{74}$$

Such that its space part coincides with the three-dimensional operator ∇, while the time part is $\partial/\partial x^0$. By applying the chain rule for derivatives, it can be seen that

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\nu}} \tag{75}$$

but from (65)

$$x^{\nu} = (\Lambda^{\mu}_{\nu})^{-1}\, x'^{\mu} \tag{76}$$

and so

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = (\Lambda^{\mu}_{\nu})^{-1} \tag{77}$$

and

$$\frac{\partial}{\partial x'^{\mu}} = (\Lambda^{\mu}_{\nu})^{-1}\,\frac{\partial}{\partial x^{\nu}} \tag{78}$$

thus, according to (60), the components of the 4-gradient transform covariantly. Hence the differentiation with respect to the contravariant components gives a covariant vector operator represented as ∂_{μ} [7]

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = (\frac{\partial}{\partial x^0}, \nabla) = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla) \tag{79}$$

[7] This is not true in the general case of tensor analysis where g_{ij} is a function of the position.

The corresponding contravariant operator is given by

$$\partial^\mu = g^{\mu\nu}\partial_\nu = \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial x_0}, -\nabla) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \tag{80}$$

The 4-gradient increases the rank of the tensor by one. For example, if $\Phi(x^\mu)$ is a scalar function, its 4-gradient

$$\partial_\mu \Phi \tag{81}$$

is a covariant vector. The increment $d\Phi$ of a scalar function can be written as the product of the infinitesimal vector dx^μ and the 4-gradient $\partial_\mu \Phi$

$$d\Phi = \partial_\mu \Phi dx^\mu \tag{82}$$

Similarly, if A^μ is a contravariant vector, $\partial_\nu A^\mu$, $\partial^\nu A^\mu$ and $\partial_\nu A_\mu$ are mixed, contravariant, and covariant tensors, respectively, of second rank.

The scalar product of the differential operator ∂_μ (or ∂^μ) and the contravariant (or covariant) form of a given 4-vector is an invariant

$$\partial_\mu A^\mu = \partial^\mu A_\mu = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \vec{A} = \frac{\partial A_0}{\partial x_0} - \nabla \cdot (-\vec{A}) \tag{83}$$

which is called the 4-dimensional divergence of the 4-vector.

Taking the scalar product of (79) and (80), i.e. the 4-divergence of the 4-gradient, we get

$$\Box = -\partial_\mu \partial^\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{84}$$

where $\Box$ is the D'Alembertian eperator. The D'Alembertian operator is an invariant and does not change the rank of the tensor on which it acts.

By differentiating a 4-vector, for example $A^\mu = (A^0, \vec{A})$, we can form an antisymmetric tensor

$$T^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{85}$$

which is called the 4-dimensional curl of A^μ. Its space part ($\mu, \nu = 1, 2, 3$) is identical to the components of $\nabla \times \vec{A}$.

0.8 4-dimensional velocity and acceleration

As shown in (40), proper time $d\tau$ is an invariant, and so the derivative of any 4-vector with respect to $d\tau$ is also a 4-vector. In particular, the derivative of the coordinates of a world point $x^\mu = \left(x^0, \vec{r}\right)$ is a 4-vector u^μ, termed 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{86}$$

where the components of u^μ are given by

$$u^\mu = \left(\frac{c}{\sqrt{1 - u^2/c^2}}, \frac{\vec{u}}{\sqrt{1 - u^2/c^2}}\right) \tag{87}$$

The square of the 4-velocity is

$$u_\mu u^\mu = c^2 \tag{88}$$

From (59) and (87) we have

$$\begin{aligned}
\frac{u'_x}{\sqrt{1-u'^2/c^2}} &= \frac{1}{\sqrt{1-\beta^2}}\frac{u_x - v}{\sqrt{1-u^2/c^2}} \\
\frac{u'_y}{\sqrt{1-u'^2/c^2}} &= \frac{u_y}{\sqrt{1-u^2/c^2}} \\
\frac{u'_z}{\sqrt{1-u'^2/c^2}} &= \frac{u_z}{\sqrt{1-u^2/c^2}} \\
\frac{1}{\sqrt{1-u'^2/c^2}} &= \frac{1}{\sqrt{1-\beta^2}}\frac{1-u_x v/c^2}{\sqrt{1-u^2/c^2}}
\end{aligned} \tag{89}$$

By differentiating the 4-velocity with respect to $d\tau$, we obtain the 4-acceleration a^μ

$$a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} \tag{90}$$

and from (88) we have

$$u_\mu a^\mu = 0 \tag{91}$$

Thus the vectors u^μ and a^μ are always orthogonal. It is easy to show that the components of the 4-acceleration are

$$a^\mu = \left(\frac{\vec{u}\cdot\vec{a}}{c\left(1-u^2/c^2\right)^2}, \frac{\vec{a}}{1-u^2/c^2} + \frac{\vec{u}\,(\vec{u}\cdot\vec{a})}{c^2\left(1-u^2/c^2\right)^2} \right) \tag{92}$$

the invariant value of the square of the 4-acceleration being

$$a_\mu a^\mu = \frac{\left(\frac{\vec{u}}{c}\times\vec{a}\right)^2 - a^2}{\left(1-u^2/c^2\right)^3} \tag{93}$$

0.9 Relativistic energy and momentum

The concept of momentum can be extended from its nonrelativistic 3-dimensional formulation

$$\vec{p} = m_0\vec{u} \tag{94}$$

to a relativistic 4-dimensional covariant formulation by multiplying the 4-velocity by the scalar invariant rest mass m_0, i.e. the mass of a particle measured in its proper frame. As a result, we obtain another 4-vector, the 4-energy-momentum, or for short, 4-momentum p^μ,

$$p^\mu = m_0 u^\mu = \left(\frac{m_0 c}{\sqrt{1-u^2/c^2}}, \frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}} \right) = m\,(c,\vec{u}) = (E/c,\vec{p}) \tag{95}$$

where

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \tag{96}$$

represents the mass of the particle; this is no longer a constant since its value depends on the velocity of the particle in a given frame. The mass of the particle simplifies to m_0 in its proper frame. The space part, $\vec{p}$, of p^μ

$$\vec{p} = \frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}} = m\vec{u} \tag{97}$$

is the vector momentum of the particle which, for $u << c$, simplifies to its nonrelativistic definition, $\vec{p} = m_0 \vec{u}$. The quantity E in (95)

$$E = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} = mc^2 \tag{98}$$

represents the relativistic energy of the particle. Expression (98) is called the mass-energy relationship. Taking into account (95) and (98) and that the square of the 4-momentum, which is invariant, is given by $p^\mu p_\mu = m_0^2 c^2$ we get the relativistic formula relating the energy E, momentum p and rest mass m_0 of a particle

$$E^2 = m_0^2 c^4 + p^2 c^2 = \left(m_0 c^2\right)^2 + p^2 c^2 \tag{99}$$

The term, $m_0 c^2$ is called the rest energy and is a constant that does not depend on whether the object is moving or not, nor on whether any external field is acting on the particle. Particularly in the frame where $u = 0$, that is $p = 0$, (98) reduces to the famous formula

$$E = m_0 c^2$$

The kinetic energy is the difference between the relativistic and the rest energy

$$T = E - m_0 c^2 = m_0 c^2 \left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right)$$

At the nonrelativistic limit, $u << c$, $\left(1 - u^2/c^2\right)^{-\frac{1}{2}}$ can be expanded in powers of u^2/c^2

$$\left(1 - u^2/c^2\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + ... \tag{100}$$

and the relativistic energy of the particle simplifies to

$$E \simeq m_0 c^2 (1 + \frac{u^2}{2c^2}) = m_0 c^2 + \frac{m_0 u^2}{2} \tag{101}$$

where

$$T = \frac{m_0 u^2}{2}. \tag{102}$$

is the nonrelativistic kinetic energy of the particle. For relativistic velocities, such that $E >> m_0 c^2$, $E \simeq pc$.

0.9.1 The 4-vector Minkowski force

To obtain a covariant version of Newton's second law

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{103}$$

where $\vec{F}$ is the ordinary 3-dimensional force, we define, as a natural extension of (103), the 4-vector Minkowski force K^μ as

$$K^\mu = \frac{dp^\mu}{d\tau} = \frac{d\left(m_0 u^\mu\right)}{d\tau} = m_0 a^\mu = \left(K^0, \vec{K}\right) \tag{104}$$

or, from (42), in terms of ordinary time

$$K^\mu = \frac{1}{\sqrt{1-u^2/c^2}} \frac{d\left(m_0 u^\mu\right)}{dt} \tag{105}$$

The space components, $\vec{K}$, of K^μ are related to the 3-dimensional force $\vec{F}$, by

$$\vec{K} = \frac{1}{\sqrt{1-u^2/c^2}} \vec{F} \tag{106}$$

The time component of (104)

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c} \frac{dE}{d\tau} \tag{107}$$

represents the proper rate, divided by c, at which the energy of the particle changes.

From the scalar product of (104) by the 4-velocity, we get

$$K^\mu u_\mu = m_0 a^\mu u_\mu = 0 \tag{108}$$

and from (87), (104) and (106) we have

$$K^\mu u_\mu = K^0 \frac{c}{\sqrt{1-u^2/c^2}} - \frac{1}{(\sqrt{1-u^2/c^2})^2} \vec{F} \cdot \vec{u} = 0 \tag{109}$$

so that

$$K^0 = \vec{F} \cdot \vec{u} \frac{1}{c\sqrt{1-u^2/c^2}} = \frac{1}{c\sqrt{1-u^2/c^2}} \frac{dW}{dt} \tag{110}$$

where

$$\vec{F} \cdot \vec{u} = \frac{dW}{dt} \tag{111}$$

which expresses that the variation in energy per unit of time is equal to the work of the force on a particle per unit of time. Therefore, the component K^0 is related to the work of the three-dimensional force $\vec{F}$.

From (105), (106) and (87) we have

$$\begin{aligned}\vec{F} &= \frac{d}{dt}\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}} = \frac{m_0}{\sqrt{1-u^2/c^2}}\frac{d\vec{u}}{dt} + \frac{\vec{u}}{c^2}\frac{d}{dt}\frac{m_0c^2}{\sqrt{1-u^2/c^2}} \\ &= \frac{m_0}{\sqrt{1-u^2/c^2}}\vec{a} + \frac{\vec{u}}{c^2}\frac{dE}{dt} \qquad (112)\end{aligned}$$

and, hence, taking into account (111), we get

$$\vec{a} = \frac{\sqrt{1-u^2/c^2}}{m_0}(\vec{F} - \frac{\vec{u}}{c^2}\vec{F}\cdot\vec{u}) \qquad (113)$$

which is the covariant expression of the ordinary three-dimensional acceleration of a particle of rest mass m_0 and velocity $\vec{u}$ under the influence of an external force $\vec{F}$.

Covariant formulation of Maxwell's equations. Field transformation.

Unlike Newtonian mechanics, classical electrodynamics is consistent with special relativity and consequently Maxwell's equations are invariant. Our objective now is to write the fundamental equations of the electromagnetic field in covariant form by means of tensor relations in the 4-dimensional space. This provides two advantages: on the one hand, the purely formal advantage that a simpler and more symmetric formulation is thus achieved, and on the other that new and fundamental connections appear between the different physical magnitudes of the electromagnetic field theory. This is essential to achieve a profound understanding of electromagnetic processes.

0.10 4-Vector current density

To write the laws of electromagnetism in covariant form, we take as a basic postulate the empirally proven hypothesis that the charge carried by a particle, for example an electron or a proton, is a fundamental magnitude that does not vary with the particle's velocity and has the same value in all inertial frames. Of course this can be extended to the charge possessed by a given quantity of matter which is the sum of all its subatomic particles. Consequently, the electrical charge, unlike mass, is invariant under Lorentz transformation. Mathematically, this property is expressed by means of the charge continuity equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \vec{u}) + \frac{\partial \rho}{\partial t} = 0 \tag{114}$$

which can be written in covariant form defining the contravariant current 4-vector J^{μ} as

$$J^{\mu} = \left(c\rho, \vec{J} \right) = \rho \left(c, \vec{u} \right) \tag{115}$$

where ρ is the charge density and $\vec{u}$ is the velocity at which the charge distribution moves. Hence the continuity equation takes the concise covariant form

$$\partial_\mu J^\mu = 0 \tag{116}$$

which states that the 4-current density has no divergence.

To see how the definition of the 4-vector J^μ implies the charge conservation, let us consider a charge distribution ρ_0 at rest in frame S. In this system, the components of the 4-dimensional current vector J^μ are

$$J^0 = c\rho_0, \ \vec{J} = 0 \tag{117}$$

On changing from inertial reference system S to S', the 4-vector J^μ transforms according to (50). Therefore, in S' the components of the 4-vector J'^μ are

$$\begin{aligned} J'^0 &= c\rho_0\gamma = c\rho' \\ J'^1 &= -v\rho_0\gamma \\ J'^2 &= 0 \\ J'^3 &= 0 \end{aligned} \tag{118}$$

from which it follows that the charge density in movement $\rho' = \rho_0\gamma$ is greater than the charge density at rest ρ_0 by a factor γ. Thus, the charge Q contained in a given volume is invariant, because of the transformation law of a volume V_0 moving at velocity $u = v$ is

$$V = \frac{V_0}{\gamma} \tag{119}$$

and so

$$Q = \rho' V = \rho_0 V_0 = Q_0 \tag{120}$$

which means that the total charge is an invariant.

0.11 Covariant formulation of wave equations for potentials

We can write the wave equations for the potentials Φ and $\vec{A}$,

$$\nabla^2 \vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = \Box \vec{A} = -\mu_0 \vec{J} \tag{121a}$$

$$\nabla^2 \Phi - \mu_0\varepsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = \Box \Phi = -\frac{\rho}{\varepsilon_0} \tag{121b}$$

in covariant form by introducing the 4-vector potential A^μ

$$A^\mu = \left(A^0, A^1, A^2, A^3\right) = \left(\Phi/c, \vec{A}\right) \tag{122}$$

where $\vec{A}$ is the 3-vector potential. Thus, Eqs. (121) take the compact covariant form

$$\Box A^{\mu} = -\mu_0 J^{\mu} \tag{123}$$

which, according to the properties of the D'Alembertian operator clearly shows that (122) is a 4-vector.

The Lorenz condition

$$\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial \Phi}{\partial t} = 0 \tag{124}$$

can be written as the 4-dimensional divergence of A^{μ} taking the simplified form

$$\partial_{\mu} A^{\mu} = 0 \tag{125}$$

Hence $\partial_{\mu} A^{\mu}$ is a Lorentz scalar and thus, the Lorenz gauge is covariant under transformation in special relativity. The coulomb gauge $\nabla \cdot \vec{A} = 0$ is not Lorentz invariant.

0.12 Covariant formulation of Maxwell's equations

We can now express the components of the electromagnetic field in covariant form. Let us start from the expressions of the 4-vector potential, (122), together with those of the electromagnetic field in terms of the potentials

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \tag{126a}$$

$$\vec{B} = \nabla \times \vec{A} \tag{126b}$$

These two equations can be expressed in covariant form using the 4-dimensional curl, (85), of the 4-potential, to obtain a contravariant antisymmetric tensor $F^{\mu\nu}$ of rank 2, called the electromagnetic field tensor,

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{127}$$

or in matrix form

$$F^{\mu\nu} = \begin{matrix} & \nu \longrightarrow \\ \mu \downarrow & \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \end{matrix} \tag{128}$$

such that the elements of the tensor are comprised of the components of the electromagnetic field.

Maxwell's equations

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \tag{129a}$$
$$\nabla \times \vec{B} = \mu_0(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) \tag{129b}$$

can be unified and written in covariant form as the 4-divergence of $F^{\mu\nu}$ such that

$$\partial_\nu F^{\mu\nu} = \partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu = \begin{bmatrix} \nabla \cdot \vec{E}/c \\ \nabla \times \vec{B}\Big|_x - \frac{1}{c^2}\frac{\partial E_x}{\partial t} \\ \nabla \times \vec{B}\Big|_y - \frac{1}{c^2}\frac{\partial E_y}{\partial t} \\ \nabla \times \vec{B}\Big|_z - \frac{1}{c^2}\frac{\partial E_z}{\partial t} \end{bmatrix} = \mu_0 J^\mu = -\Box A^\mu \tag{130}$$

where we have taken into account (123) and (125).

The remaining two Maxwell's equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$

can be expressed as

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{131}$$

where $F_{\mu\nu}$ is the covariant field tensor obtained from $F^{\mu\nu}$ by index lowering using the metric tensor $g_{\alpha\beta}$, (69), such that,

$$F_{\mu\nu} = g_{\mu\lambda} g_{\nu\delta} F^{\lambda\delta} = \mu \downarrow \overset{\nu \longrightarrow}{\begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}} \tag{132}$$

To put Eq. (131) in a more compact form, we first introduce the completely antisymmetric 4-rank tensor $\epsilon^{\alpha\beta\mu\nu}$ defined by the following conditions

$$\epsilon^{\alpha\beta\mu\nu} = \begin{cases} 0 & \text{if any two indexes, } \alpha\beta\mu\nu, \text{ are equal} \\ +1 & \text{if the indexes } \alpha\beta\mu\nu \text{ form an even permutation of 0123} \\ -1 & \text{if the indexes } \alpha\beta\mu\nu \text{ form an odd permutation of 0123} \end{cases} \tag{133}$$

such that only $4! = 24$ components of $\epsilon^{\alpha\beta\mu\nu}$ are *non*-0. It is easy to show that $\epsilon^{\alpha\beta\mu\nu}$ is an invariant tensor. The *dual tensor* $G^{\mu\nu}$ of $F_{\mu\nu}$ is defined by the relation

$$G^{\mu\nu} = \tfrac{1}{2}c\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \begin{matrix} & \nu \longrightarrow \\ \mu\downarrow & \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{bmatrix} \end{matrix} . \tag{134}$$

Note that $G^{\mu\nu}$ can be directly obtained from $F^{\mu\nu}$ by changing $\vec{E}/c$ to $c\vec{B}$ and $\vec{B}$ to $-\vec{E}$. Using the dual tensor the equation (131) can be writtten as

$$\partial_\mu G^{\mu\nu} = 0 \tag{135}$$

Since $F^{\mu\nu}$ is a 4-tensor we can find the expressions for the transformation of the components of the electromagnetic field from

$$F'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta} \tag{136}$$

Hence we can easily obtain

$$E'_x = E_x \quad B'_x = B_x \tag{137a}$$

$$E'_y = \gamma(E_y - vB_z) \quad B'_y = \gamma(B_y + \tfrac{v}{c^2}E_z) \tag{137b}$$

$$E'_z = \gamma(E_z + vB_y) \quad B'_z = \gamma(B_z - \tfrac{v}{c^2}E_y) \tag{137c}$$

or in compact form

$$\vec{E}' = \vec{E}_\parallel + \gamma(\vec{E}_\perp + \vec{v}\times\vec{B}) \tag{138a}$$

$$\vec{B}' = \vec{B}_\parallel + \gamma(\vec{B}_\perp - \frac{1}{c^2}\vec{v}\times\vec{E}) \tag{138b}$$

In these expressions, the subscripts $\parallel$ and $\perp$ indicate the components that are parallel and perpendicular, respectively, to the relative motion of the coordinate systems. It is evident from (137), or (138), that it is possible to transform magnetic into electric fields and vice versa by changing the frame, and that the electric or magnetic field is zero in a reference system S and non-zero in S'. Hence, it is not possible to distinguish absolutely between electric and magnetic fields. For low velocities, $v << c$, (138) simplifies to

$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B} \qquad \vec{B}' = \vec{B} - \tfrac{1}{c^2}\vec{v}\times\vec{E} \tag{139}$$

As ε_0 and μ_0 are invariant, we can write (138) alternatively as

$$\vec{D}' = \vec{D}_\parallel + \gamma(\vec{D}_\perp + \vec{v}/c^2\times\vec{H}) \tag{140a}$$

$$\vec{H}' = \vec{H}_\parallel + \gamma(\vec{H}_\perp - \vec{v}\times\vec{D}) \tag{140b}$$

which for low velocities simplifies to

$$\vec{D}' = \vec{D} + \tfrac{1}{c^2}\vec{v}\times\vec{H} \qquad \vec{H}' = \vec{H} - \vec{v}\times\vec{D} \tag{141}$$

0.12.1 Field invariants

It is useful to know which electromagnetic quantities remain invariant under Lorentz transformations. To this end, we take into account that just as the scalar product of two 4-vectors, $A^\mu B_\mu$, is a scalar, so in general is the multiplication of two 4-tensors when each upper index is summed with a lower index. Thus, using the field tensor components, we have the following invariants

$$F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - E^2/c^2\right) \tag{142}$$

$$F_{\mu\nu}G^{\mu\nu} = \frac{1}{2}c\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = 4\left(\vec{B}\cdot\vec{E}\right) \tag{143}$$

and therefore[8],

$$c^2B^2 - E^2 = inv \tag{144}$$

$$\vec{B}\cdot\vec{E} = inv \tag{145}$$

From the invariance of these quantities, we can easily draw the following conclusions concerning the behaviour of the electromagnetic field under the Lorentz transformation:

1) If $c^2B^2 > E^2$ and $\vec{B}\perp\vec{E}$ in a reference frame S, then there is another coordinate system S' where $E = 0$ and $B \neq 0$ (or vice versa if $c^2B^2 < E^2$)

2) If in a reference frame S there exists only the electric field, or only the magnetic field, in another reference frame there will be both fields, but they will be mutually orthogonal.

3) If the angle $\theta = \cos^{-1}(\vec{E}\cdot\vec{B}/EB)$ is acute, or obtuse, in a reference frame S it will remain acute, or obtuse, respectively, in any other inertial frame.

4) For plane electromagnetic waves, definedby $cB = E$ and $\vec{B}\perp\vec{E}$, both invariants vanish and the wave remains a plane wave in any other inertial frame. Nevertheless, the frequency and direction of the waves will vary with the frame of reference as we will see *later* when we study the Doppler effect.

0.13 Lorentz force density 4-vector

It is easy to show that the spatial components of the 4-vector

$$f^\mu = F^{\mu\nu}J_\nu \tag{146}$$

is the Lorentz force density $\vec{f}$

$$\vec{f} = \rho\left(\vec{E} + \vec{u}\times\vec{B}\right) = \rho\vec{E} + \vec{J}\times\vec{B} \tag{147}$$

while the time component is

$$f^0 = \rho\frac{\vec{u}\cdot\vec{E}}{c} = \frac{\vec{J}\cdot\vec{E}}{c} = \frac{\vec{u}\cdot\vec{f}}{c} \tag{148}$$

[8] Of course we could try to obtain another invariant from $G_{\mu\nu}G^{\mu\nu}$ but the result would be the same invariant (144)

This is equal to $1/c$ times the power per unit volume delivered to the charge distribution by the electromagnetic field (no work is done by the magnetic field because its force is orthogonal to the velocity). Hence we have

$$f^{\mu} = F^{\mu\nu} J_{\nu} = \rho \left(\frac{\vec{u} \cdot \vec{E}}{c}, \vec{E} + \vec{u} \times \vec{B} \right) = \left(\frac{\vec{J} \cdot \vec{E}}{c}, \rho\vec{E} + \vec{J} \times \vec{B} \right) \tag{149}$$

The 4-vector f^{μ} is usually referred to as the Lorentz force density 4-vector.

0.14 4-Dimensional Minkowski force vector

To get a covariant expression for the net electromagnetic force acting on the charge contained in a 3-dimensional spatial volume element dV, we cannot simply multiply the density of force 4-vector f^{μ} by dV because, as we know, dV is not an invariant, its value depending on the reference-frame velocity according to, (22),

$$dV = \frac{dV'}{\gamma} \tag{150}$$

Instead, we can multiply f^{μ} by the 4-dimensional volume element d^4x,

$$d^4x = dx^0 dx^1 dx^2 dx^3 = dx^0 dV \tag{151}$$

which is invariant under the Lorentz transformation, as can be easily understood taking into account that, while the volume of ordinary 3-space is reduced by a factor γ, the time is dilated by the same factor, (23), $dt = \gamma dt'$. Hence

$$d^4x' = d^4x \tag{152}$$

and thus the Lorentz transformation preserves the space-time volume[9] d^4x. Using the 4-dimensional vector of force density, (146), and the invariance of the 4-dimensional volume, we can construct a 4-dimensional *vector of momentum*

$$dP^{\mu} = f^{\mu} dx_1 dx_2 dx_3 dt = f^{\mu} \frac{dx^0 dV}{c} \tag{154}$$

[9]Mathematically, we can prove the invariance of the 4-dimensional volume writing down the expression for the transformation of volume elements with change of variables

$$d^4x' = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} d^4x \tag{153}$$

where

$$\frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} = \begin{bmatrix} \frac{\partial x'^0}{\partial x^0} & \frac{\partial x'^1}{\partial x^0} & \frac{\partial x'^2}{\partial x^0} & \frac{\partial x'^3}{\partial x^0} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x'^0}{\partial x^3} & \frac{\partial x'^1}{\partial x^3} & \frac{\partial x'^2}{\partial x^3} & \frac{\partial x'^3}{\partial x^3} \end{bmatrix} = 1$$

is the Jacobian of the transformation *which can be* easily seen *from* (47) is the unity.

Dividing both sides of this equation by the invariant element of proper time of the particle $d\tau$ and integrating with respect to the space coordinates, we obtain the 4-dimensional force vector acting on the particle

$$K^{\mu} = \int_V \frac{dP^{\mu}}{d\tau} = \int_V f^{\mu} dV \frac{dt}{d\tau} = \int_V f^{\mu} \frac{dV}{\sqrt{1-u^2/c^2}} \tag{155}$$

The 4-dimensional force K^{μ} is called the Minkowski force.

For a point particle, we have

$$\int \rho dV = q \tag{156}$$

Hence

$$K^{\mu} = \left(\frac{q}{c\sqrt{1-u^2/c^2}} \left(\vec{u} \cdot \vec{E} \right), \frac{\vec{F}}{\sqrt{1-u^2/c^2}} \right) \tag{157}$$

where $\vec{F}$, is the Lorentz force

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) \tag{158}$$

The time component K^0 of (157)

$$K^0 = \frac{q}{\sqrt{1-u^2/c^2}} \frac{\vec{E} \cdot \vec{u}}{c} \tag{159}$$

is equal to $1/c$ times the proper power delivered to the particle by the electromagnetic field.

It is easy to show that, in terms of $F^{\mu\nu}$ and of the 4-velocity u^{μ}, the Minkowski force on a charge q is given by

$$K^{\mu} = q u_{\nu} F^{\mu\nu} \tag{160}$$

0.15 4-Dimensional energy-momentum field tensor

It is possible to express the Lorentz force density f^{μ} (149) as the 4-divergence of a tensor $T^{\mu\beta}$, i.e.

$$f^{\mu} = -\partial_{\beta} T^{\mu\beta} \tag{161}$$

To find the expression of $T^{\mu\beta}$ we start from (130)

$$\partial_{\nu} F^{\mu\nu} = \mu_0 J^{\mu} \tag{162}$$

which, lowering the index μ on both sides, can be written as

$$\partial_{\nu} F_{\mu}^{\nu} = \mu_0 J_{\mu} \tag{163}$$

so that

$$J_\mu = \frac{1}{\mu_0}\partial_\nu F^\nu_\mu \tag{164}$$

Thus

$$f^\mu = F^{\mu\nu}J_\nu = \frac{1}{\mu_0}F^{\mu\nu}\partial_\beta F^\beta_\nu = \frac{1}{\mu_0}\left(\partial_\beta(F^{\mu\nu}F^\beta_\nu) - F^\beta_\nu\partial_\beta F^{\mu\nu}\right) \tag{165}$$

where the last term on the right can be expressed as

$$\begin{aligned} F^\beta_\nu\partial_\beta F^{\mu\nu} &= F_{\beta\nu}\partial^\beta F^{\mu\nu} = F_{\nu\beta}\partial^\nu F^{\mu\beta} = F_{\beta\nu}\partial^\nu F^{\beta\mu} = \\ \frac{1}{2}\left(F_{\beta\nu}\partial^\nu F^{\beta\mu} + F_{\nu\beta}\partial^\nu F^{\mu\beta}\right) &= \frac{1}{2}F_{\beta\nu}\left(\partial^\nu F^{\beta\mu} + \partial^\beta F^{\mu\nu}\right) = -\frac{1}{2}F_{\beta\nu}\partial^\mu F^{\nu\beta} = \\ \frac{1}{2}F_{\beta\nu}\partial^\mu F^{\beta\nu} &= \frac{1}{4}\partial^\mu\left(F^{\beta\nu}F_{\beta\nu}\right) = \frac{1}{2}\partial^\mu(B^2 - E^2/c^2) \end{aligned} \tag{166}$$

In steps 2 and 5, the dummy indexes ν, β were interchanged, and in steps 3 and 7 we made use of the antisymmetric character of the tensor $F^{\beta\nu}$. Furthermore, in steps 6 and 9 we used (131) and (142), respectively. Equations (165) and (166) can be combined to give

$$\begin{aligned} f^\mu &= F^{\mu\nu}J_\nu = \frac{1}{\mu_0}\left(\partial_\beta(F^{\mu\nu}F^\beta_\nu) - \frac{1}{2}g^{\mu\beta}\partial_\beta(B^2 - E^2/c^2)\right) = \\ & \frac{1}{\mu_0}\partial_\beta\left(F^{\mu\nu}F^\beta_\nu - \frac{1}{2}g^{\mu\beta}(B^2 - E^2/c^2)\right) \end{aligned} \tag{167}$$

Hence, from (161)

$$T^{\mu\beta} = \frac{1}{\mu_0}\left(\frac{1}{2}g^{\mu\beta}(B^2 - E^2/c^2) - F^{\mu\nu}F^\beta_\nu\right) \tag{168}$$

which is a symmetric tensor called the energy-momentum field tensor. Using (128) we can write the components of the energy-momentum tensor in terms of the electromagnetic field as

$$T^{\mu\beta} = \quad \mu\downarrow \overset{\beta\longrightarrow}{\begin{bmatrix} W & \mathcal{P}_x/c & \mathcal{P}_y/c & \mathcal{P}_z/c \\ cg_x & T^{em}_{11} & T^{em}_{12} & T^{em}_{13} \\ cg_y & T^{em}_{21} & T^{em}_{22} & T^{em}_{23} \\ cg_z & T^{em}_{31} & T^{em}_{32} & T^{em}_{33} \end{bmatrix}} \tag{169}$$

since the indexes $\mu, \beta = 1, 2, 3$ correspond to the coordinates x, y, z. Thus the 9 elements

$$T^{\mu\beta} : \mu, \beta = 1, 2, 3; \tag{170}$$

coincide with the Maxwell stress tensor $T^{em}_{\beta\alpha}$, defined in (**??**) and $\vec{\mathcal{P}}$ and $\vec{g} = \vec{\mathcal{P}}/c^2$ are the Poynting and density of momentum respectively, such that

$$c\vec{g} = \frac{\vec{\mathcal{P}}}{c} \tag{171}$$

as would be expected, because it is clear from (168) that the tensor $T^{\mu\beta}$ is symmetric. The scalar W

$$W = \frac{1}{2}(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}) \tag{172}$$

is, according to (**??**) and (**??**), the energy density of the electromagnetic field in free space.

To clarify the meaning of the various components of $T^{\mu\beta}$ let us begin by integrating the time part of (161) over the 3-dimensional volume $dV = dxdydz$

$$\int f^0 dV = -\int \frac{1}{c}\frac{\partial W}{\partial t} dV - \int \nabla \cdot \frac{\vec{\mathcal{P}}}{c} dV \tag{173}$$

where f^0 is defined in (148). This equation is identical to (**??**) and therefore represents the energy-conservation equation for the electromagnetic field. This is the reason to use the Poynting's vector $\vec{P}$ in the first row of (169).

Regarding the spatial part of (161), it is easily shown that these reduce to

$$\frac{\partial \vec{g}}{\partial t} + \nabla \cdot T^{em}_{\beta\alpha} = -\rho\vec{E} - \vec{J} \times \vec{B} \tag{174}$$

which is, according to (**??**), the momentum-conservation equation for the electromagnetic field.

0.16 Doppler effect

Let us consider a plane harmonic wave defined by

$$\begin{aligned} \vec{E} &= \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \\ \vec{H} &= \vec{H}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \end{aligned} \tag{175}$$

We know that the invariants (144) and (145) ensure that the character of the wave is invariant. Furthermore, the fact that $\vec{E}$ and $\vec{H}$ are simultaneously null at a given point in space and time, i.e., at a world point, is absolute and thus must be fulfilled in any inertial system. This means that the phase of the wave has the same value in all coordinate systems and is invariant, i.e.,

$$\varphi = \omega t - \vec{k} \cdot \vec{r} = inv \tag{176}$$

This invariant can be written as the scalar product of two 4-vectors

$$\varphi = k^\mu x_\mu \tag{177}$$

where k^μ is the wave 4-vector

$$k^\mu = \left(\frac{\omega}{c}, \vec{k}\right) \tag{178}$$

with $\vec{k}$ being the ordinary wave vector and x_μ the covariant form of the 4-vector coordinates of a world-point. From (177) it is evident that k^μ is 4-vector.

If $\hat{n}$ is the unit vector in the propagation direction of the plane wave then

$$\vec{k} = \frac{\omega}{c}\hat{n} \tag{179}$$

and we can rewrite (178) as

$$k^\mu = \frac{\omega}{c}\,(1, \hat{n}) = \frac{\omega}{c}\,(1, n_x, n_y, n_z) \tag{180}$$

with n_x, n_y and n_z being the projection of the unit vector $\hat{n}$ onto the x, y and z axes (the director cosines of the wave vector $\vec{k}$).

Let us consider a light source moving at velocity v with respect to an inertial system S and let S' be a proper system, (Figure 25.1). By using the Lorentz transformations (50) we obtain from (180) the following expressions for transforming the frequency and direction of a plane wave

$$\omega n_x = \omega' \gamma\,(\beta + n'_x)\,; \qquad \omega n_y = \omega' n'_y \tag{181a}$$

$$\omega n_z = \omega' n'_z; \qquad \omega = \gamma\omega'\,(1 + \beta n'_x) \tag{181b}$$

Therefore, assuming that the light source moves at velocity v, the frequency perceived by the observer at rest is

$$\omega = \gamma\omega'\,(1 + \beta n'_x) = \gamma\omega'\,\left(1 + \beta\cos\theta'\right) \tag{182}$$

where we have introduced the angle θ' between the propagation direction $\hat{n}'$ and the axis x' by determining $n'_x = \cos\theta'$. The relation (182) explains the change in the colour of the light emitted by a moving source, as perceived by an observer at rest. This frequency change due to the relative movement of the source and the receiver is known as the Doppler effect or the Doppler shift. Of course, the transformation of the frequency between system S and S' is

$$\omega' = \gamma\omega\,(1 - \beta n_x) \tag{183}$$

This formula enables us to determine the observed frequency as a function of the frequency f' of the light emitted in the proper system S' by a light source and of $n_x = \cos\theta$, with θ being the angle between the direction of the light $\hat{n}$ and the x axis in the reference system.

If the direction of the wave propagation coincides with that of the movement, we have $n'_x = n_x = \pm 1$, where the value $+1$ corresponds to the source moving towards the observer, and the value -1 corresponds to the source moving away from the observer. Thus, from (183) we have the longitudinal Doppler effect

$$\omega = \frac{\omega'}{(1 \mp \beta)\,\gamma} \tag{184}$$

which for $\beta << 1$ ($\gamma \simeq 1$, non-relativist velocities), simplifies to

$$\omega \simeq \omega'\,(1 \pm \beta) \tag{185}$$

This equation coincides with the classical expression for the Doppler effect.

When the emission is $n_x = 0$, i.e. the light source is moving at right angles to a line from the observer to the source, we get the so-called transverse Doppler shift

$$\omega = \omega' \sqrt{1-\beta^2} \simeq \omega' \left(1 - \frac{1}{2}\beta^2\right);\ \beta << 1 \tag{186}$$

This is a second-order effect, since the dominant term is proportional to β^2, in contrast to the ordinary Doppler shift (182) which is of first order in β. According to Newtonian physics, this transversal Doppler shift should not exist in this case because such a shift is an exclusively relativistic effect associated with the time-dilation factor γ. The transversal Doppler effect was experimentally measured by Ives and Stilwellin 1938 . The Ives-Stilwell experiment constitute together the Michelson-Morley one, a fundamental validation of the theory of relativity.

0.17 Plane waves.

0.17.1 Transformation of amplitude

Let there be a plane wave defined by

$$\vec{E} = \vec{E}_0 e^{j\varphi};\ \vec{H} = \vec{H}_0 e^{j\varphi} \tag{187}$$

where the wave vector is $\vec{k} = (\omega/c)\hat{n}$ and the electric field is on the xy plane (Fig 5). The field components are

$$E_x = -E_0 n_y e^{j\varphi} \tag{188a}$$
$$E_y = E_0 n_x e^{j\varphi} \tag{188b}$$
$$H_z = \frac{1}{\eta_0} E_0 e^{j\varphi} \tag{188c}$$

From the field-transformation relations (137),

$$E'_x = E_x \tag{189a}$$
$$E'_y = \gamma\left(E_y - v\mu_0 H_z\right) \tag{189b}$$
$$H'_z = \gamma\left(H_z - v\varepsilon_0 E_y\right) \tag{189c}$$

taking into account the invariance of the phase φ

$$E'_0 n'_y = E_0 n_y \tag{190a}$$
$$E'_0 n'_x = \gamma E_0\left(n_x - \beta\right) \tag{190b}$$
$$E'_0 = \gamma E_0\left(1 - \beta n_x\right) \tag{190c}$$

Dividing the second equation of (183) by (190c), we obtain

$$\frac{\omega'}{E'_0} = \frac{\omega}{E_0} = inv \tag{191}$$

In other words, the amplitude to frequency ratio in a plane wave is invariant and, thus, its value does not depend on the inertial reference system.

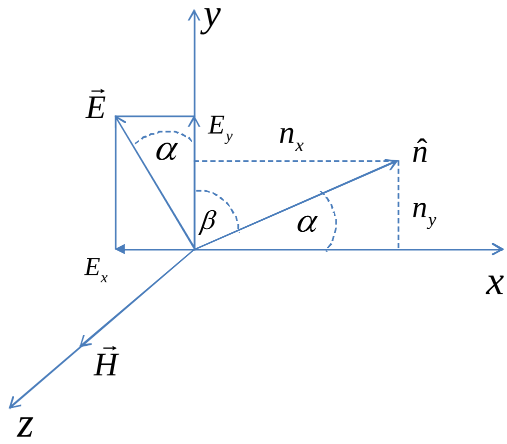

Figure 5:

0.18 Energy-momentum 4-vector of a plane wave

It is well known that the energy associated with a plane wave in a volume in space V, within reference system S, is given by

$$W = \frac{1}{2}\int_V \left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right) dV = \int_V \varepsilon_0 E_0^2 dV \tag{192}$$

Since the wave moves at the velocity of light, so does V. To determine the effects on V of the change from S to S', we introduce an auxiliary volume V_0 moving at a velocity $\vec{u}$ with respect to S and at a velocity $\vec{u}'$ with respect to S'. According to the Lorentz contraction, for this volume, we have

$$V' = V_0\left(1 - \frac{u'^2}{c^2}\right)^{\frac{1}{2}} \; ; \quad V = V_0\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}} \tag{193}$$

By dividing, we get

$$\left(\frac{V'}{V}\right)^2 = \frac{1 - u'^2/c^2}{1 - u^2/c^2} \tag{194}$$

Since in the present case, $u' = u = c$, the above equation needs to be particularized for these limit values. Assuming u and u' in the direction of the x axis and taking into account the transformation of velocities (89)

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2} \tag{195}$$

and, after substituting in (194), we find

$$\frac{V'}{V} = \frac{1}{\gamma\left(1 - vu_x/c^2\right)} \tag{196}$$

Since $u_x = \vec{u} \cdot \hat{n}_x$, and u tends to c, we have

$$V' = \frac{V}{\gamma \left(1 - \beta n_x\right)} \tag{197}$$

·and from (190c)

$$V' = \frac{V E_0}{E_0'} \tag{198}$$

Given that $E_0/\omega = E_0'/\omega' = inv$, we obtain

$$V' = V \frac{\omega}{\omega'}; \quad V'\omega' = V\omega \tag{199}$$

from which we may conclude that $V\omega$ is another invariant.

From the above and from (192), we have

$$\frac{W}{\omega} = \int_V \varepsilon_0 \frac{E_0^2}{\omega} dV = \frac{W'}{\omega'} = inv \tag{200}$$

or, alternatively,

$$\frac{W}{f} = \frac{W'}{f'} \tag{201}$$

That is, the energies measured by different observers of a train of plane waves are related to each other in the same way as the frequencies are for these observers.

Let us consider the moment associated with a plane wave. As well as with energy, a plane wave in a volume V is associated with a moment equal to

$$\vec{G} = \frac{1}{c^2} \int_V \vec{E} \times \vec{H} dV = \frac{\hat{n}}{c} \int_V \varepsilon_0 E_0^2 dV = \hat{n} \frac{W}{c} \tag{202}$$

where $\hat{n}$ is the unit vector in the direction of propagation.

By means of the invariant W/ω, we can write

$$\vec{G} = \hat{n}\omega \frac{W}{c\omega} = \hat{n}\omega d; \quad d = \frac{W}{c\omega} \tag{203}$$

where d is invariant. Now, multiplying the wave 4-vector k^μ defined in (180) by the invariant cd, we get the energy-momentum 4-vector G^μ of a train of plane waves

$$G^\mu = cdk^\mu = \frac{W}{\omega} k^\mu = \frac{W}{\omega} \frac{\omega}{c} \left(1, \hat{n}\right) = \left(\frac{W}{c}, \vec{G}\right) \tag{204}$$

where W and $\vec{G}$ are the total moment and energy.

Arbitrarily moving point charge fields.

0.19 1.- The Liénard Wiechert potentials.

To find the electromagnetic field due to an arbitrarily moving point charge q, it is easier to calculate previously the covariant form of the retarded potentials. To simplify the task, we use the procedure of obtaining first the covariant form of the potentials in an instantaneous proper frame, i.e., a frame in which the particle is at rest at a given instant of time and then generalize the result for any other inertial frame. [10]

Let the coordinates of the point charge (source point) defined by the position 4-vector be $x'^{\mu} = (ct', \vec{r}\,')$ and let the coordinates of the point at which the potential is evaluated (field point) be

$$x^{\mu} = (ct, \vec{r}_0) \tag{205}$$

see figure tal. The field and source points define the 4-vector

$$R^{\mu} = x^{\mu} - x'^{\mu} = (c(t - t'), \vec{r}) \tag{206}$$

where $\vec{r} = \vec{r}_0 - \vec{r}\,'$. As the electromagnetic field created by the particle propagates from x'^{μ} to x^{μ} at the velocity of c, we have

$$r = c(t - t') \tag{207}$$

and

$$R^{\mu} = (r, \vec{r}) \tag{208}$$

so that

$$R^{\mu} R_{\mu} = c^2 (t - t')^2 - r^2 = 0 \tag{209}$$

In the coordinate system in which the particle is instantaneously at rest, there exists only the Coulomb potential, and the 4-potential (122) simplifies to

$$A^{\mu} = \left(A^0, A^1, A^2, A^3\right) = \left(\frac{\Phi}{c}, \vec{0}\right) = \left(\frac{q}{4\pi\varepsilon_0 r c}, \vec{0}\right) \tag{210}$$

[10] En los capitulos que siguen llamaremos t' en vez de τ al tiempo retardado para no confundirlo con el tiempo propio

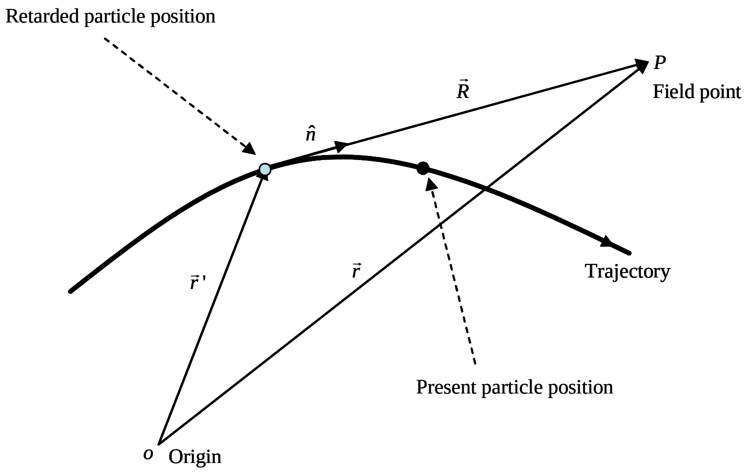

Figure 6: Trajectory, retarded position, present position, and field point of an accelerated charge.

To obtain an expression for the 4-potential in a coordinate system in which the particle is moving arbitrarily, we must rewrite (210) as a 4-vector relationship. For this, we note, from (87) that in its proper frame the 4-velocity u^μ of the particle is

$$u^\mu = \left(\frac{c}{\sqrt{1-u^2/c^2}}, \frac{\vec{u}}{\sqrt{1-u^2/c^2}} \right)_{u=0} = (c, \vec{0}). \tag{211}$$

Hence we can write

$$A^\mu = \left(\frac{qu^\mu}{4\pi\varepsilon_0 rc^2} \right) \tag{212}$$

According to (208) and (211) the invariant $u^\nu R_\nu$ is

$$u^\nu R_\nu = rc \tag{213}$$

and therefore for the 4-potential we get the covariant expression

$$A^\mu = \left(\frac{qu^\mu}{4\pi\varepsilon_0 cu^\nu R_\nu} \right) \tag{214}$$

subject to the condition (209). Since equation (214) is relativistically invariant and valid in any inertial system it can be generalized to a coordinate system in which the particle is moving by simply taking into account that in general $u \neq 0$, and then $u^\nu R_\nu = (rc - \vec{u}\cdot\vec{r})\,/\sqrt{1-u^2/c^2}$. Thus

$$A^\mu = \frac{q}{4\pi\varepsilon_o c}\frac{(c,\vec{u})}{rc - \vec{r}\cdot\vec{u}} = \left(\Phi/c, \vec{A}\right) \tag{215}$$

where

$$\Phi(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{1}{s} \quad (216a)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_o q}{4\pi}\frac{\vec{u}}{s} = \frac{\vec{u}}{c^2}\Phi(\vec{r},t) \quad (216b)$$

and s

$$s = \left(r - \frac{\vec{r}\cdot\vec{u}}{c}\right) = r\left(1 - \frac{\hat{n}\cdot\vec{u}}{c}\right); \qquad \hat{n} = \vec{r}/r \quad (217)$$

is a function of both, the field point and the source point. The expressions (216a) and (216b), valid for any charge velocity, are the so-called Liénard-Wiechert potentials. It should be noted that the velocity $\vec{u}$ of the charged particle and the radiovector $\vec{r}$ from the position of the particle to the point at which the potential is evaluated must be taken not at time t but at $t' = t - r/c$. When $u << c$ $(u/c \rightarrow 0)$ then (215) tends to (210).

0.20 Arbitrarily moving point charge fields

The fields can be calculated from the potentials in the usual manner,(**??**) and (**??**),

$$\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \nabla\times\vec{A}$$

where Φ and $\vec{A}$ are given by (216a), (216b). Thus, assuming that we know the position of the charge as a function of the time, we can calculate the electric and magnetic fields observed at position $\vec{r}_0$ and time t due to an arbitrarily moving charge q for which the retarded position and time are $\vec{r}'$ and t', respectively *(see Figure tal)*. To calculate the fields, we must first transform $\partial/\partial t$ and ∇, using the chain rule for derivatives

$$\begin{aligned}\vec{E} &= -\nabla\Phi - \frac{\partial\vec{A}}{\partial t} = -\frac{q}{4\pi\varepsilon_o}\nabla\left(\frac{1}{s}\right) - \frac{\mu_0 q}{4\pi}\frac{\partial}{\partial t}\left(\frac{\vec{u}}{s}\right)\\ &= \frac{q}{4\pi\varepsilon_o s^2}\nabla s - \frac{q\mu_0}{4\pi s}\left(\frac{\partial\vec{u}}{\partial t} - \frac{\vec{u}}{s}\frac{\partial s}{\partial t}\right) \quad (218)\\ \vec{B} &= \nabla\times\vec{A} = \frac{\mu_o q}{4\pi}\nabla\times\frac{\vec{u}}{s} = \frac{\mu_o q}{4\pi}\left[\frac{\nabla\times\vec{u}}{s} + \nabla\left(\frac{1}{s}\right)\times\vec{u}\right] \quad (219)\end{aligned}$$

In these expressions the components of the gradient operator ∇ are the partial derivatives at constant observation time t but not at constant retarded time t'. However, the trajectory, velocity and acceleration of the charge are given with respect to the retarded time t'. Thus we must transform (218) and

(219) into expressions involving the operators $\partial/\partial t'$ and $\nabla' = \nabla|_{t'=cte}$. To this end, we use the following transformation for the derivatives

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}; \qquad t' = t - r(t')/c \tag{220}$$

where

$$\frac{\partial t'}{\partial t} = \frac{\partial(t - r/c)}{\partial t} = 1 - \frac{1}{c}\frac{\partial r}{\partial t'}\frac{\partial t'}{\partial t} \tag{221}$$

and thus

$$\frac{\partial t'}{\partial t} = \frac{1}{1 + \frac{1}{c}\frac{\partial r}{\partial t'}} \tag{222}$$

The quantity $\partial r/\partial t'$ can be found taking into account that

$$r\frac{\partial r}{\partial t'} = \vec{r}\cdot\frac{\partial \vec{r}}{\partial t'} = -\vec{r}\cdot\vec{u} \tag{223}$$

therefore

$$\frac{\partial r}{\partial t'} = -\frac{\vec{r}\cdot\vec{u}}{r} = -\hat{n}\cdot\vec{u} \tag{224}$$

where $\vec{r}/r = \hat{n}$ (see Figure tal..). Inserting (224) in (222) we have

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \hat{n}\cdot\vec{u}/c} = \frac{r}{s} \tag{225}$$

and therefore

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \frac{r}{s}\frac{\partial}{\partial t'} \tag{226}$$

Furthemore since $r = c(t - t')$, we have

$$\nabla t' = \nabla\left(t - \frac{r}{c}\right) = -\frac{\nabla r}{c} = -\frac{1}{c}\left(\nabla' r + \frac{\partial r}{\partial t'}\nabla t'\right) \tag{227}$$

hence

$$\nabla t' = -\frac{\vec{r}}{c\,(r - \vec{r}\cdot\vec{u}/c)} = -\frac{\vec{r}}{cs} \tag{228}$$

and, in general

$$\nabla = \nabla' - \frac{\vec{r}}{cs}\frac{\partial}{\partial t'} \tag{229}$$

Equations (226) and (229) are the basic transformation rules from the coordinates of the field point to those of the retarded field point. These rules allow us to compute the derivatives of the Liénard-Wierchert potentials to obtain

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\left[\frac{r}{c^2 s^3}\frac{\partial s}{\partial t'}\vec{u} - \frac{r}{c^2 s^2}\vec{a} + \frac{1}{s^2}\nabla' s - \frac{1}{cs^3}\vec{r}\frac{\partial s}{\partial t'}\right] \tag{230a}$$

$$\vec{B} = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{rc}\left(\vec{r}\times\frac{r}{c^2 s^3}\frac{\partial s}{\partial t'}\vec{u} - \vec{r}\times\frac{r}{c^2 s^2}\vec{a} + \frac{r}{cs}\nabla'\times\vec{u} - \frac{r}{cs^2}\nabla' s\times\vec{u}\right)\right] \tag{230b}$$

where $\vec{a} = d\vec{u}/dt'$ is the charge acceleration at t'. Expressions (230a) and (230b) can be simplified, observing that

$$\begin{aligned} \frac{\partial s}{\partial t'} &= \frac{\partial}{\partial t'}\left(r - \frac{\vec{r}\cdot\vec{u}}{c}\right) = \frac{\partial r}{\partial t'} - \frac{\partial \vec{r}}{\partial t'}\cdot\frac{\vec{u}}{c} - \frac{\vec{r}}{c}\cdot\vec{a} \\ &= -\frac{\vec{r}\cdot\vec{u}}{r} + \frac{u^2}{c} - \frac{\vec{r}\cdot\vec{a}}{c} \end{aligned} \tag{231}$$

and

$$\nabla' s = \frac{\vec{r}}{r} - \frac{1}{c}\nabla'(\vec{r}\cdot\vec{u}) = \frac{\vec{r}}{r} - \frac{\vec{u}}{c} \tag{232}$$

To obtain (232) we must take into account that when deriving while keeping t' constant, the vector $\vec{u}$ remains constant. After combining the terms containing $\vec{a}$ to form a triple vector product, we get

$$\vec{E} = \frac{q}{4\pi\varepsilon_o s^3}\left[\left(\vec{r} - \frac{\vec{u}r}{c}\right)\left(1 - \frac{u^2}{c^2}\right) + \frac{1}{c^2}\vec{r}\times\left(\left(\vec{r} - \frac{r\vec{u}}{c}\right)\times\vec{a}\right)\right] \tag{233}$$

A similar calculation yields

$$\vec{B} = \frac{\mu_o q}{4\pi}\left[\nabla\times\frac{\vec{u}}{s}\right] = \frac{\mu_o q}{4\pi}\left[\frac{\nabla\times\vec{u}}{s} + \nabla\left(\frac{1}{s}\right)\times\vec{u}\right] \tag{234}$$

In these expressions, $\vec{a}$, $\vec{u}$ and $\vec{r}$ must be evaluated at retarded time t'.

Moreover, as $u = u(t')$, we have

$$(\nabla\times\vec{u})_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial t'}\frac{\partial t'}{\partial y} - \frac{\partial u_y}{\partial t'}\frac{\partial t'}{\partial z} = -\left[\vec{a}\times\nabla t'\right]_x \tag{235}$$

and taking into account (228)

$$\nabla\times\vec{u} = -\left[\vec{a}\times\nabla t'\right] = -\frac{\vec{r}\times\vec{a}}{cs} \tag{236}$$

Finally, we obtain

$$\vec{B} = \frac{\mu_o q}{4\pi}\left[-\frac{\vec{r}\times\vec{a}}{cs^2} - \frac{1}{s^2}\left(\nabla s\times\vec{u}\right)\right] \tag{237}$$

which, using (233), can be written as

$$\vec{B} = \frac{1}{c}\left[\frac{\vec{r}}{r}\times\vec{E}\right] = \frac{1}{c}\left[\hat{n}\times\vec{E}\right] \tag{238}$$

Thus the magnetic field is always perpendicular to $\vec{E}$ and to the vector $\vec{r}$.

The electric field (233) created by the charge can be divided into two parts, $\vec{E} = \vec{E}_i + \vec{E}_{rad}$, where

$$\vec{E}_i = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{s^3}\left(\vec{r} - \frac{\vec{u}r}{c}\right)\left(1 - \frac{u^2}{c^2}\right)\right] \tag{239}$$

is the induction field, which depends solely on the velocity of the charge and varies spatially as r^{-2}.

The other component, the radiation field $\vec{E}_{rad}$, depends on the velocity and on the acceleration and decreases at large distances from the charge as r^{-1} and so at such distances $E \simeq E_{rad}$. However at short distances and ¿ ¿low frequencies??, the induction field predominates on the radiation field.

The radiation field

$$\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_o c^2}\left[\frac{1}{s^3}\left\{\vec{r} \times \left(\left(\vec{r} - \frac{r\vec{u}}{c}\right) \times \vec{a}\right)\right\}\right] \tag{240}$$

The radiation field is always perpendicular to $\vec{r}$ and $\vec{a}$ (transversal field) . Due to the inverse factor c^{-2}, large acceleration values are needed to produce significant radiation. According to (238), the radiation magnetic field is given by

$$\vec{B}_{rad} = \frac{1}{c}\left[\hat{n} \times \vec{E}_{rad}\right] \tag{241}$$

This shows that the magnetic radiation field from the particle is always perpendicular to its electric field and the position vector from its retarded position. When $\vec{u}$ and $\vec{a}$ are null the electric field reduces to that created by a static electric charge, and the magnetic field becomes null.

The fact that a particle moving at a uniform velocity cannot radiate is consistent with the relativistic nature of fields, since there exists a proper reference system in which no energy is radiated because the particle is at rest.

Electrons may reach high velocities within certain devices (such as betatrons and synchrotrons), where they are confined by magnetic fields and forced to move in circular orbits so that the electrons radiate energy. The velocity increases to the point at which the energy is radiated as the same rate as it is supplied by the accelerator. Thus, in practice, there is an upper limit to the energy that electrons can reach within a circular accelerator. This limit lies within the energy range $10-20$ Bev ($1Bev = 10^9 eV$). Furthermore, the radiation produced by the linear acceleration of electrons (which is a low value compared with that produced by a circular movement) does not have such a low upper limit. Linear accelerators can produce energy levels of 100 to 300 Bev. The radiation of energy by protons moving in circular orbits is much less than for electrons and thus circular devices are the most economical way of accelerating protons at very high energy levels.

La energía también se radia por deceleración. Por tanto si se proyecta un haz de electrones sobre un bloque de material que los frene se emitirá radiación. Es este caso, la radiación es llamada radiación X o "bremstrahlung" y, así , precisamente se producen los rayos X.

0.21 Fields from a charged particle in uniform motion

If the velocity $\vec{u}$ of the particle is constant i.e. $\vec{a} = 0$, the fields can be expressed in terms of the present position of the particle.

In the case of a charge moving at a constant velocity, $\vec{u}$, it is possible to express $\vec{E}$ and $\vec{B}$ as functions of present rather than of retarded values. To see this, consider Fig. 27.3. In this figure, $aa' = vR/c$ is the distance travelled by the charge in time r/c; r is the distance from the charge to the filed point P at the retarded position, and r_o is the distance from the charge to the field point at the present position. Clearly,

$$s = \left(r - \frac{\vec{u} \cdot \vec{r}}{c} \right) = r - \frac{ur}{c} \cos\theta = NP$$

From the rectangular triangle ANP we see that

$$s^2 = r_o^2 - aN^2 = r_o^2 - \frac{u^2 r^2 \sin^2\theta}{c^2}$$

Moreover, $OP = r \sin\theta$. This implies that

$$s = r_o \left(1 - \beta^2 \sin^2\theta_o\right)^{\frac{1}{2}}$$

Taking into account that $\vec{r}_o = \vec{r} - (\vec{u}r/c)$, we can write for $\vec{E}$, from (27.22)

$$\overrightarrow{E} = \frac{q}{4\pi\varepsilon_o} \vec{r}_o \left(1 - \beta^2\right) \frac{1}{r_o^3 \left(1 - \beta^2 \sin^2\theta_o\right)^{\frac{3}{2}}}$$

Thus the electric field points from the present source point to the field point, which is somewhat unexpected. The electric field lines are straight and come from the present position of the particle. The faster the particle moves, the greater is the density of the electric field lines in the direction perpendicular to that of the movement.

Then the electric field is pointing in the direction from the present source point to the field point. This is a little unexpected result. The electric field lines are straight lines coming from the present position of the particle. The pattern follows the particle. They become denser and denser in the direction perpendicular to the direction of motion the faster the particles moves.

In a radial direction, moving away from the instantaneous position of the point charge. However, unlike the static case, it is not strictly symmetric. In fact, for a fast-moving charge, the field is concentrated in the plane perpendicular to its movement. The magnetic field is arranged in circles centred on the trajectory of the charge.

All the magnitudes stated in the above relations are evaluated at the present instant. Moreover, from $\vec{B} = \vec{r} \times \vec{E}/c$ we obtain

$$B = \frac{E}{c} \sin\widehat{(a'Pa)}$$

and by applying the sine law in the triangle $a'Pa$

$$\vec{B} = \frac{\vec{u} \times \vec{E}}{c^2}$$

Obviously, from (27.29) and (27.30) for high velocities ($\beta \to 1$), the fields increase in the direction perpendicular to that of the movement. Thus, when $u \to c$, the total field begins to resemble a plane wave, although no radiation exists, due to the fact that the dependence of the fields is determined by $1/r^2$.

Figure 27.4 shows the distribution of the electric field for an electric charge in uniform movement for different values of β.

Radiation of energy and moment by accelerated particles

0.22 Introduction

In this chapter, we will find the general expression for the energy and moment radiated by an accelerated charged particle. To do so, we use the results from the previous chapter concerning the fields created by a charge under arbitrary movement. Next, we will particularize these expressions for the case in which the movement of the particle is due to the effect of an electromagnetic field. We then calculate the braking radiation, which is emitted as a consequence of a beam of charged particles passing through a Coulomb field that creates static charges. Finally, we study the angular distribution of the radiated energy.

0.23 Angular distribution of the energy radiated by an accelerated charge

The rate of energy loss for an accelerated charge per unit of solid angle $d\Omega$ is

$$\frac{dP(t')}{d\Omega} = \frac{dW}{dt'd\Omega} = \frac{dW}{dtd\Omega}\frac{dt}{dt'} = \frac{dP(t)}{d\Omega}\frac{s}{r} \tag{242}$$

where we have taken into account that, according to (225), $\partial t/\partial t' = s/r$. Using (240) and (241) to calculate the Poynting vector and remembering the definition of differential of solid angle ($d\Omega = \hat{n} \cdot d\vec{s}/r^2$), we get the general expression for the power radiated per unit of solid angle in the frame of reference of the particle (i.e. $u = v$),

$$\frac{dP(t')}{d\Omega} = rs\vec{\mathcal{P}}(t) \cdot \hat{n} = \frac{q^2}{16\pi^2\varepsilon_o c^3}\frac{1}{r^4}\frac{\left(\vec{r} \times [(\vec{r} - r\vec{u}/c) \times \vec{a}]\right)^2}{\left(1 - \hat{n} \cdot \vec{u}/c\right)^5} \tag{243}$$

Although this expression is in general complex, it is easy to see that there is no radiation in the direction in which $\vec{a} \times (\vec{r} - r\vec{u}/c)$ is annulled. In the case of

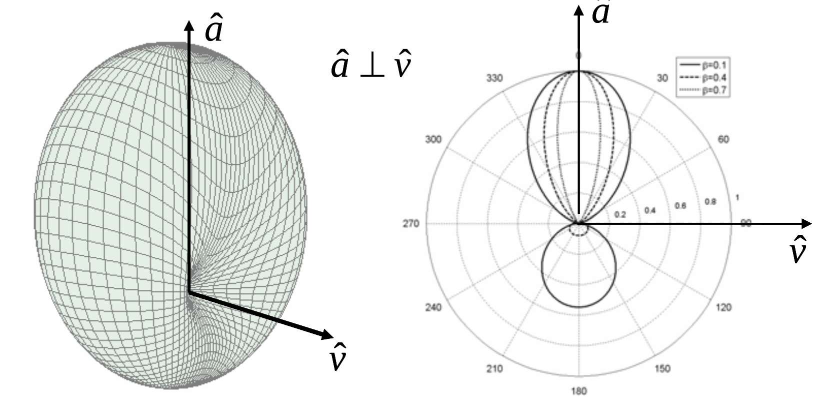

Figure 7:

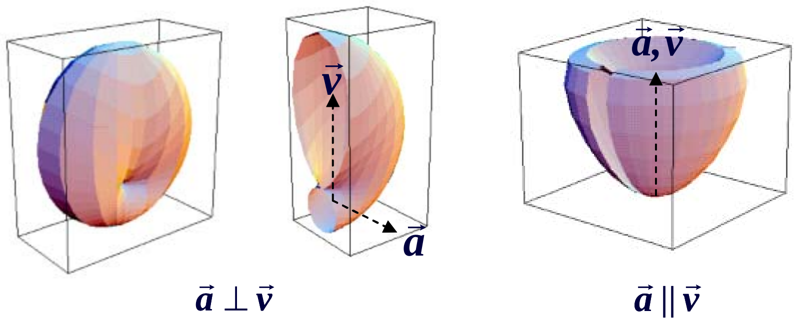

Figure 8:

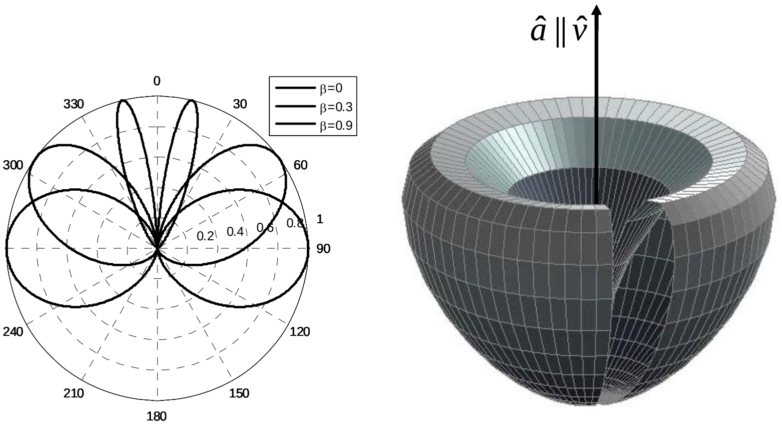

Figure 9:

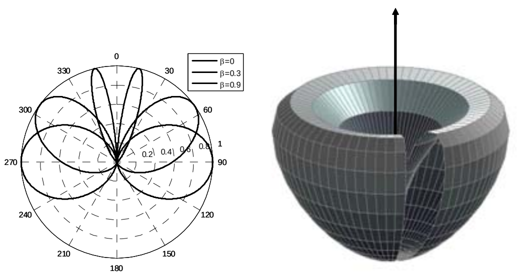

Figure 10: $\beta = 0,6$ para la de la derecha

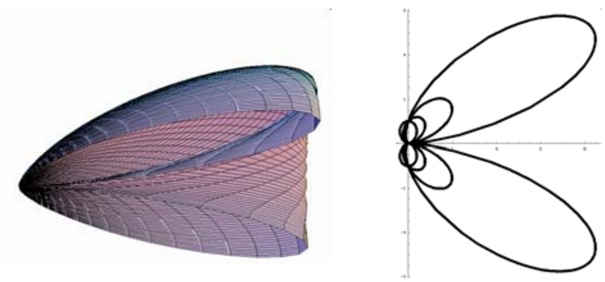

Figure 11: .8 de v/c y .25 a 3

velocities close to the speed of light, the radiation intensity (243) is high, due to the high exponential power of $(1-\hat{n}\cdot\vec{u}/c)$ in the denominator, within the narrow angular interval in which $(1-\hat{n}\cdot\vec{u}/c)$ is small. Thus, the particle radiates mainly in the direction of its movement. Let us now consider two particular cases:

1) Velocity and acceleration are parallel. It is straightforward to see that

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_o c^3}\left[\frac{a^2\sin^2\theta}{(1-u\cos\theta/c)^5}\right] \tag{244}$$

where θ is the angle formed by $\vec{a}$ *(or* $\vec{u}$) and $\hat{n}$. The radiation diagram for this case is shown in Figure Tal.

The power radiated is dependent on the magnitude squared of the acceleration. However, as the angle does not depend on the acceleration. It depends on the speed. The faster the particle moves the more is the radiation concentrated in the forward direction. This is opposite to the nonradiative fields for particles in uniform motion, where the field strengths were increasingly concentrated in the perpendicular direction.

For $u \ll c$ the distribution acquires the functional form of the dipolar radiation $\sin^2\theta$, but for relativistic velocities, the maxima are strongly forward-tilted. The large exponent of the denominator means that for ultrarelativistic particles ($\gamma >> 1$) and for values of $\cos\theta \approx 1$ the denominator is very small. Consequently, almost all the radiation is concentrated within the region of small θ but there is no radiation for $\theta = 0$.

The total radiated power can be determined from the expression (244) integrating over every direction, thus obtaining

$$P(t') = \frac{q^2a^2\gamma^6}{6\pi\varepsilon_o c^3} \tag{245}$$

The factor γ^6 means that the radiated energy increases enormously as the charge (or particle) velocity approaches the speed of the light.

An example of applying (244) is the calculation of the radiation, called "braking radiation" or "bremsstrahlung", which occurs during the deceleration of charged particles passing through a material medium, as for example when a high-speed electron strikes a metal target. For an exact calculation, we need to know the dependence of deceleration on time; however, for an approximate calculation, we can assume that $\vec{a}$ is constant while velocity decreases from the value u_o to zero. Thus (244) becomes

$$dW = \frac{q^2a^2}{16\pi^2\varepsilon_o c^3}\int_{u_0}^{0}\frac{\sin^2\theta dt' d\Omega}{(1-u\cos\theta/c)^5} \tag{246}$$

$$\frac{dW}{d\Omega} = -\frac{q^2\sin^2\theta a^2}{64\pi^2\varepsilon_0 c^2\cos\theta}\left[\frac{1}{(1-u_0\cos\theta/c)^4}-1\right] \tag{247}$$

This equation is usually used to estimate the efficiency of a low-voltage X-ray tube.

2) Acceleration is perpendicular to the velocity. By developing the numerator of (243), we get

$$\frac{dP(t')}{d\Omega} = \frac{q^2 a^2}{16\pi^2 \varepsilon_o c^3} \frac{1}{(1 - u\cos\theta/c)^3} \left[1 - \frac{\sin^2\theta \cos^2\varphi}{\gamma^2 (1 - u\cos\theta/c)^2} \right] \tag{248}$$

where θ is again the angle formed by $\hat{n}$ and $\vec{u}$, while φ is the azimuthal angle of the vector $\hat{n}$ with respect to the plane containing $\vec{u}$ and $\vec{a}$,as shown in Figure 28.5. As in the case of bremsstrahlung, the velocity causes the radiation pattern to tilt in the forward direction, i.e. towards the direction of velocity.

The radiation diagram presents a maximum that is mainly in the direction of velocity $\vec{u}$ such that, each time the velocity of the particle is towards a static observer, radiation pulses would be seen.

Figure 28.6 shows the radiation diagrams in the plane of the orbit ($\varphi = 0$) for some values of $\beta = u/c$. The dashed lines represent the directions of zero intensity.

The total power can be calculated by integrating the expression (248).

$$P(t') = \frac{q^2 a^2 \gamma^4}{6\pi\varepsilon_o c^3} \tag{249}$$

A comparison of (245) and (249) shows that for the same magnitude of force applied, the power radiated for linear acceleration is γ^2 times greater than that of transversal acceleration. Therefore, for relativistic particles accelerating in an arbitrary direction, the effect of acceleration in the direction of the movement predominates (in terms of energy loss by radiation) over the effect of the instantaneous component of circular movement. Note that, since the angular distribution depends on a squared, the radiation does not change whether the particle is under acceleration or deceleration and, in both cases, the radiation is focused on the velocity-forward direction.

0.24 Energy and the moment of radiation of accelerated charges

From expressions (240) and (241) the radiation fields created by a particle in accelerated movement are given by

$$\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_o s^3 c^2} \left\{ \vec{r} \times \left[\left(\vec{r} - \frac{r\vec{u}}{c} \right) \times \vec{a} \right] \right\} \tag{250a}$$

$$\vec{B}_{rad} = \frac{\hat{n} \times \vec{E}_{rad}}{c} \tag{250b}$$

Let us consider an instantaneous rest frame S where the charge is at rest at the instant of radiation, i.e. its velocity is zero but not its acceleration. In this case, the radiation fields and the Poynting vector take the following form

$$\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_o c^2 r^3}\vec{r}\times(\vec{r}\times\vec{a}) \tag{251a}$$

$$\vec{B}_{rad} = \frac{q}{4\pi\varepsilon_o c^3 r^2}(\vec{a}\times\vec{r}) \tag{251b}$$

$$\vec{\mathcal{P}}_{rad} = \frac{q^2}{16\pi^2\varepsilon_o c^3}\frac{\vec{r}\,(\vec{r}\times\vec{a})^2}{r^5} \tag{251c}$$

To find the total radiated power, we must integrate the Poynting vector over a closed surface surrounding the charge. We may determine this surface to be a sphere centred on the retarded position of the charge. By setting the z axis in the direction of $\vec{a}$ as shown in Figure *tal (la 28.1 del libro en español)*, we have

$$P_{rad} = -\frac{dW_0}{dt} = \int_S \vec{\mathcal{P}}_{rad}\cdot\hat{n}dS = \frac{q^2\left[\vec{a}^2\right]}{6\pi\varepsilon_o c^3} \tag{252}$$

where P_{rad} is the radiated power and W_0 the energy of the particle in its proper system. The minus sign in (252) indicates that the charge loses energy.

It is noteworthy that the Eqs. (251a) and (251b) are formally identical to those obtained for the radiation fields of an electric dipole of dipolar moment $\vec{d} = \vec{a}q/\omega^2$. Its radiation diagram is therefore as in Figure 28.2. As a result of the acceleration, considering for example the case of an electron, we find that it will lose energy by radiation at a rate given by

$$\frac{dW_0}{dt} = -\frac{e^2\left[a^2\right]}{6\pi\varepsilon_o c^3} \tag{253}$$

Moreover, due to the dependence on $\sin^2\theta$ of the dipolar radiation, the fields emitted in opposing directions are equal in absolute value in the proper frame. Thus, the net momentum transported by the field is zero

$$\frac{d\vec{G}_0}{dt} = 0 \tag{254}$$

We can now write Eqs. (253) and (254) in covariant form. To do so, we should bear in mind that in the system in which the particle is at rest at a given instant, $d\tau = dt$ is fulfilled.

Let us consider a wave train radiated during the interval $d\tau$ by the electron, with transported energy and momentum dW_0 and dG_0. The energy and the momentum of a finite wave train constitute the quadrivector

$$G^\mu = \left(\frac{W}{c}, \vec{G}\right) \tag{255}$$

In the proper system of the particle, we have

$$G^\mu = \left(\frac{W_0}{c}, \vec{0}\right) \tag{256}$$

and

$$u^\mu = \left(c, \vec{0}\right) \quad ; \quad a^\mu = (0, \vec{a}) \tag{257}$$

so that

$$a^2 = -a^\mu a_\mu \tag{258}$$

With these equations, we have

$$\frac{dG^\mu}{d\tau} = -\frac{e^2 a^2}{6\pi\varepsilon_o c^5} u^\mu \tag{259}$$

which can be written in covariant form as

$$\frac{dG^\mu}{d\tau} = \frac{e^2 a^\mu a_\mu}{6\pi\varepsilon_o c^5} u^\mu \tag{260}$$

Thus, in the special case in which the charge is instantaneously at rest in the coordinate system, the radiation of the electron is correctly expressed by (260), in accordance with (253) and (254). Now, given that (260) has a covariant form, we can conclude that it also describes the radiation in another coordinate system in which the electron is moving at an arbitrary velocity. Using the relations (42) and (93) and the 4-velocity expression (87), we obtain from (260) the following general expressions for the changes, due to radiation, in the energy W and in the momentum $\vec{G}$ of the electron

$$\frac{dW}{dt} = \frac{e^2}{6\pi\varepsilon_o c^3} \frac{\left(\frac{\vec{u}}{c} \times \vec{a}\right)^2 - \vec{a}^2}{\left(1-\beta^2\right)^3} = -\frac{e^2}{6\pi\varepsilon_o c^3} \frac{a^2\left(1-\beta^2\right) + \left(\vec{u}\cdot\vec{a}\right)^2/c^2}{\left(1-\beta^2\right)^3} \tag{261a}$$

$$\frac{d\vec{G}}{dt} = \frac{e^2 \vec{u}}{6\pi\varepsilon_o c^5} \frac{\left(\frac{\vec{u}}{c} \times \vec{a}\right)^2 - a^2}{\left(1-\beta^2\right)^3} = -\frac{e^2 \vec{u}}{6\pi\varepsilon_o c^5} \frac{a^2\left(1-\beta^2\right) + \left(\vec{u}\cdot\vec{a}\right)^2/c^2}{\left(1-\beta^2\right)^3} \tag{261b}$$

This enables us to find, in an arbitrary coordinate system, the energy and momentum of the radiation field created by a charge under accelerated movement, assuming the movement equation to be known. Clearly, these expressions are reduced to (253) and (254) when $u \to 0$.

The accelerated movement of very fast particles is frequently the result of the action of an electromagnetic field. The relativist expression for the acceleration is (113)

$$\vec{a} = \left(1-\beta^2\right)^{\frac{1}{2}} \frac{1}{m_0}\left[\vec{F} - \frac{\vec{u}}{c^2}(\vec{F}\cdot\vec{u})\right] \tag{262}$$

where m_0 is the static mass of the particle and $\vec{F}$ is the force acting upon it. Assuming that $\vec{F}$ is the Lorentz force, the radiation of energy per unit of time by a charge moving in an electromagnetic field is

$$\frac{dW}{dt} = -\frac{e^4}{6\pi\varepsilon_o m_0^2 c^3} \frac{\left(\vec{E} + \vec{u}\times\vec{B}\right)^2 - \frac{1}{c^2}\left(\vec{u}\cdot\vec{E}\right)^2}{1-\beta^2} \tag{263}$$

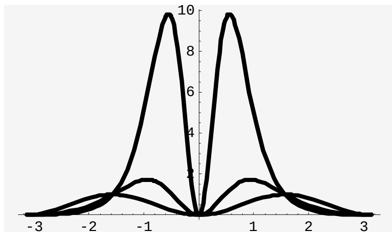

Figure 12:

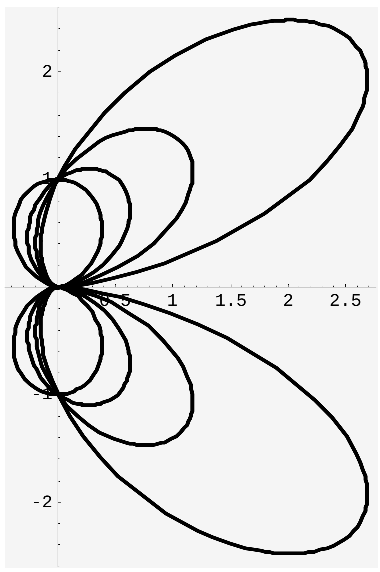

Figure 13:

Taking into account the expression for the total energy of a particle

$$W = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \tag{264}$$

we can write Eq. (263) as

$$\frac{dW}{dt} = -\frac{e^4}{6\pi\varepsilon_o m_0^4 c^7} W^2 \left[\left(\vec{E} + \vec{u} \times \vec{B} \right)^2 - \frac{1}{c^2} \left(\vec{u} \cdot \vec{E} \right)^2 \right] \tag{265}$$

Let us now consider some particular cases of (265).

1. When $\vec{B} = 0$, two cases can be distinguished:

   (a) If $\vec{E}$ is perpendicular to the velocity $\vec{u}$, then

   $$\frac{dW}{dt} = -\frac{e^4}{6\pi\varepsilon_o m_0^4 c^7} E^2 W^2 \tag{266}$$

   (b) If $\vec{E}$ is parallel to $\vec{u}$, we have

   $$\frac{dW}{dt} = -\frac{e^4}{6\pi\varepsilon_o m_0^4 c^7} W^2 E^2 \left(1-\beta^2\right) = -\frac{e^4 E^2}{6\pi\varepsilon_o m_0^2 c^3} \tag{267}$$

   where the radiation is independent of the energy of the particle.

2. When the particle is moving in a constant, uniform magnetic field such that $\vec{u}$ is perpendicular to $\vec{B}$ and $\vec{E} = 0$ (under these conditions the particle would move in *circles*). Then, (265) becomes

$$\frac{dW}{dt} = -\frac{e^4}{6\pi\varepsilon_0 m_0^4 c^7} W^2 u^2 B^2 = -\frac{e^4 u^2 B^2}{6\pi\varepsilon_0 m_0^2 c^3} \frac{1}{1-\beta^2} \tag{268}$$

Expressions (266), (267), and (268) are used in nuclear physics to determine the energy loss by radiation from relativist particles moving within electric and magnetic fields. It is worth noting that, except when $\vec{B} = 0$, and $\vec{E}$ is parallel to $\vec{u}$, the radiation depends largely on the energy of the particle.

Examples of particles within a magnetic field are the charged particles of the cosmic rays in the Earth's magnetic field, and the charged particles within a betatron. The energy losses by radiation determine not only the upper energy limit for particles that may reach the Earth, but also the energy level to which particles can be accelerated within a betatron. This means that in a betatron it is impossible to raise the energy level of electrons to much higher than a few hundred MeV. In synchrotrons, however, there is virtually no limit to the energy that can be transmitted to electrons from the exterior. aclarar

0.25 Braking radiation or Bremsstrahlung

An important application of the results found in the previous section is the calculation of the so-called braking radiation, which is emitted as a consequence of a beam of charged particles interacting with the Coulomb field created by static charges.

This phenomenon is used to generate X rays, and it plays an important role in the braking mechanism of high-energy particles moving through matter. As our interest lies fundamentally in the high-energy particles, we will limit our study to this case. An electron, with a relativistic velocity, which passes close to a nucleus undergoes a very slight deviation. In fact, it can be shown that, for velocities comparable to that of light, appreciable deviations can occur only when the impact parameter ρ (which is a measure of how close the incident particle, moving in a straight line, would come to the target) is of the order e^2/mc^2. As this situation cannot be treated by classical methods, we will limit ourselves to the case in which $\rho > e^2/mc^2$. Now we can assume, in a first-order approximation, that the trajectory of the particle is rectilinear and, moreover, that its velocity is constant, this being equivalent to neglecting the longitudinal component of the electric field. If the instant at which the particle passes through the closest point to the nucleus is chosen as the time origin, then the distance from the nucleus to the electron *(Fig. 28.3 de castellano)* is approximately

$$r = \sqrt{\rho^2 + v^2 t^2} \tag{269}$$

The tranversal component of the electric field created by the nucleus is

$$E_t = \frac{Ze}{r^2}\frac{\sin\theta}{4\pi\varepsilon_0} = \frac{Ze\rho}{4\pi\varepsilon_o r^3} = \frac{Ze\rho}{4\pi\varepsilon_0\left(\rho^2 + v^2t^2\right)^{\frac{3}{2}}} \tag{270}$$

where Z is the atomic number of the nucleus. Replacing this expression in (266) and integrating over the time from $-\infty$ to ∞, we find that the total energy loss of the particle, $\triangle W$, is given by

$$\triangle W = -\frac{r_o^2 Z^2 ce^2}{16\varepsilon_0 \rho^3 v\left(1-\beta^2\right)} \tag{271}$$

where $r_o = e^2/4\pi\varepsilon_o m_0 c^2$ is the classical radius of the electron. We see that the energy loss increases rapidly as Z increases and as the impact parameter decreases. In practice, a particle may pass at any distance from a nucleus; thus, multiplying (271) by $2\pi\rho n d\rho$, where n is the density of particles in the beam and then integrating for all the values of ρ, we find that the effective radiation of a beam of particles is

$$W_{ef} = -\frac{2\pi r_o^2 Z^2 ce^2 n}{16\varepsilon_0 v\left(1-\beta^2\right)}\int_{\rho_{\min}}^{\infty}\frac{d\rho}{\rho^2} = -\frac{\pi r_o^2 Z^2 ce^2 n}{8\varepsilon_0 v\left(1-\beta^2\right)}\frac{1}{\rho_{\min}} \tag{272}$$

In this expression, $\rho_{\min} \neq 0$ is the smallest distance to which the electron may approach the nucleus. According to quantum mechanics this parameter is given by

$$\rho_{\min} = \frac{h\gamma}{mc} \tag{273}$$

and (272) becomes

$$W_{ef} = -\frac{\pi r_o^2 Z^2 c^2 n e^2 m}{8\varepsilon_o v h}\gamma \tag{274}$$

This expression enables us to calculate losses derived from *bremsstrahlung* when very fast particles pass through matter. This radiation is a fundamental factor in determining the reduction in velocity of such particles. This type of radiation is significant for electrons with energies of 100 MeV in air and of 10 MeV in lead. For heavy particles, such as protons, these energies are much greater. It should be borne in mind that it is not possible to go from expression (274) to non-relativist formulas by simply assuming $v \ll c$ because (274) was obtained by assuming $v \sim c$.